Extra Difficulty Assignment: Recursive Real Numbers

For this assignment, I want you to complete the hardest task, namely 10. However, I suggest you work all the other tasks first.

We use the following notation.

$\mathbb{N}$ = the positive integers.
$\mathbb{Q}$ = the rational numbers.
$\mathbb{R}$ = the real numbers.
$\mathbb{I} = [0,1]$ = the unit interval $\{x \in \mathbb{R} : 0 \le x \le 1\}$.

By an abuse of notation, we sometimes identify a numeral with the number it represents.

We define a language $L \subseteq \Sigma^*$, where Σ is an alphabet, to be *recursive*, or *decidable*, if some machine decides L. That machine can be implemented as a program P_L which accepts every $w \in L$, and no other strings over Σ. That is, if the input of P_L is a string $w \in \Sigma^*$, P_L must halt, and its output is 1 if $w \in L$, 0 otherwise.

We define a set of numbers $\mathbb{A} \subseteq \mathbb{N}$ to be *recursive* if the set of numerals (in any base) for members of $\mathbb{A}$ is a recursive language. A set of numbers is defined to be recursive if the set of numerals (in any given base) of members of $\mathbb{A}$ is a recursive language.

We define a real number x to be *recursive*, or *computable*, if there is a recursive function D_L such that $D_L(n)$ is the n^{th} digit of the binary[1] expansion of x.

The Rice Number of a Language

If $\mathbb{A} \subseteq \mathbb{N}$, we define $x_{\mathbb{A}} = \sum_{i \in \mathbb{A}} 2^{-i}$, the Rice number of $\mathbb{A}$. Similarly, if L is a language over a specified alphabet[2] Σ, let $w_1, w_2, \ldots, w_n, \ldots$ be the canonical order enumeration of Σ^*. Define $x_L = \sum_{w_i \in L} 2^{-i}$, the Rice number of the language L.

For example, given $\Sigma = \{a, b\}$, the Rice number of $L = \{b, aab\}$ is $2^{-3} + 2^{-9} = 0.001000001$ in binary.

Theorem 1 *The Rice number of any set of positive integers, or of any language, is in $\mathbb{I}$, and every member of $\mathbb{I}$ is the Rice number of some set of positive integers and of some language.*

Theorem 2 *A language L is recursive if and only if its Rice number is a recursive real number; similarly, a set $\mathbb{A}$ of positive numbers is recursive if and only if its Rice number is a recursive real number.*

Fractions

A *fraction* is a string of the form u/v, where u and v are numerals for integers, and v is not a numeral for zero. If f is a fraction, the value of f is $V(f) \in \mathbb{Q}$. A set $\mathbb{B} \subseteq \mathbb{Q}$ is *recursive* if the set of fractions whose values belong to $\mathbb{B}$ is a recursive language.

[1] or any base larger than 1
[2] To define the Rice number of a language, an alphabet must be specified.

Theorem 3 *Let x be any real number. Then x is recursive if and only if the question of whether a given rational number q is less than x is decidable.*

Definition 1 *A rational number q is* diadic *if $q = n/m$ for integers n, m, where m is a power of 2.*

Computable Sequences

A sequence of strings $\sigma = s_1, s_2, \ldots$ is *computable* if s_n is a recursive function of n.

Theorem 4 *If x is a recursive real number, there is a computable sequence of rational numbers which converges to x.*

The converse of Theorem 4 is false. (See Task 10 below.)

Tasks

For this assignment, complete the following tasks.

1. Prove Theorem 1.
2. Prove Theorem 2.
3. Prove Theorem 3.
4. Prove Theorem 4.
5. Prove that the Rice number of the set of even positive integers is $\frac{1}{3}$.
6. What is the Rice number of the set of positive odd inhtegers?
7. Prove that, if $\mathbb{A} \subseteq \mathbb{N}$, and $\mathbb{A}'$ is the complement of $\mathbb{A}$, then $x_{\mathbb{A}} + x_{\mathbb{A}'} = 1$
8. Prove that the Rice number of any finite subset of $\mathbb{N}$, or of any finite language, is a diadic rational number.
9. Find languages $L_1 \neq L_2$ which have the same Rice number.
10. Prove that there exists a computable sequence of rational numbers which converges to a non-recursive real number.