

Extra Difficulty Assignment: “All Languages are Decidable”

We give a (supposed) proof that all languages are decidable. We know that the proof must be incorrect, since there are undecidable languages. Your task is to find a flaw in the proof.

**Theorem 1** Every language is decidable.

*Proof:* Let $\Sigma$ be any alphabet and $L$ any language over $\Sigma$. For any integer $n$, Let $\Sigma^n$ be the set of all strings of length $n$ over $\Sigma$, a finite set of cardinality $|\Sigma|^n$. Let $L_n = L \cap \Sigma^n$, the set of all strings in $L$ of length $n$. Then $L_n$ is finite, since it cannot be bigger than $\Sigma^n$.

Every finite language is decidable. Let $P_n$ be a program which decides $L_n$. Then the following program decides $L$.

\[
\begin{align*}
\text{Read } w \in \Sigma^* \\
\text{Let } n = |w|. \\
\text{If}(P_n \text{ accepts } w) \\
\quad \text{Write 1} \\
\text{Else} \\
\quad \text{Write 0}
\end{align*}
\]

Thus, $L$ is decidable. Since $L$ is an arbitrary language, every language is decidable.

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**Ideas that do not Work**

During the exam, many students tried to find the flaw in the proof, but none did. (Although one person had a glimmer of the correct idea.) Here are some of the answers students wrote.

1. “$L_n$ is not finite.” Yes, it is, since it is a subset of the finite set $\Sigma^n$.

2. “$L_n$ is not decidable.” Yes, it is. You can find a proof that every finite language is decidable in many places on the internet.

3. “$P_n$ does not exist.” Yes, it does, since by definition, if a language is decidable, it is decided by some program.

4. “The program gets stuck at the if condition.” No it doesn’t, because a program that decides a language halts with any input.

Good Luck!