Extra Difficulty Assignment: $\mathcal{NP}$–Complete Problems

Recall that languages are equivalent to 0/1 problems, and we'll use those terms interchangeably. We have the following definitions.

Definition 1 *A language L is $\mathcal{NP}$ if it is accepted by some non-deterministic machine in polynomial time.*

Definition 2 *A language L is $\mathcal{NP}$–*complete *it is $\mathcal{NP}$, and if there is a polynomial time reduction of any given $\mathcal{NP}$ language to L.*

Theorem 1 (Cook–Levin) *Boolean satisfiability* (SAT) *is $\mathcal{NP}$–complete.*

A proof of Theorem 1 is available on the internet, but I will not present it in class: we simply accept that it is true. For all other cases, $\mathcal{NP}$–completeness of languages (0/1 problems) is derived from $\mathcal{NP}$–completeness of SAT by repeated application of Theorem 2.

Theorem 2 *If there is a polynomial time reduction of L_1 to L_2, L_2 is $\mathcal{NP}$, and L_1 is $\mathcal{NP}$–complete, then L_2 is $\mathcal{NP}$–complete.*

Proof: Let R be a $\mathcal{P}$–TIME reduction of L_1 to L_2, and let L be any $\mathcal{NP}$ language. By Definition 2, there is a $\mathcal{P}$–TIME reduction R' of L to L_1. The composition RR' is a $\mathcal{P}$–TIME reduction of L to L_2. Thus, by Definition 2, L_2 is $\mathcal{NP}$–complete. ∎

We now list some problems, each of which is $\mathcal{NP}$, and how each is proved $\mathcal{NP}$–complete by the use of Theorem 2. I also list my estimated difficulty level of finding the needed reduction as a number from 1 (trivial) to 5 (hardest).

For the project, I would like you to find the reduction of Vertex Cover to Hamiltonian Cycle. This is by far the hardest of the problems in the list. Alternatively, find one of the reductions of difficulty level 4. In any case, you must prove that your function is actually a reduction.

1. There is a $\mathcal{P}$–TIME reduction of SAT to 3-SAT, thus 3-SAT is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 4.

2. An instance of IND, the independent set problem, is an ordered pair (G, K) where G is a graph and K is an integer. There is a solution to that instance if some set I of K vertices of G is independent, that is, no two members of I are neighbors.

   There is a $\mathcal{P}$–TIME reduction of 3-SAT to IND, and thus IND is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 3.

3. An instance of the Subset Sum problem is an ordered pair (σ, K), where σ is a sequence of positive numbers and K is a number. There is a solution to that instance if there is some subsequence of σ whose sum is K.

   There is a $\mathcal{P}$–TIME reduction of IND to the Subset Sum problem, and thus the Subset Sum problem is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 4.

4. An instance of the Partition problem is a sequence σ of positive numbers. There is a solution to that instance if the total of some subsequence of σ is half the total of σ.

   There is a $\mathcal{P}$–TIME reduction of the Subset Sum problem to Partition, and thus the Partition problem is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 1.

5. An instance of Vertex Cover is an ordered pair (G, K) where G is a graph and K is an integer. There is a solution to that instance if there is a set V of K vertices of G such that every edge of G is adjacent to some member of V.

   There is a $\mathcal{P}$–TIME reduction of IND to Vertex Cover, and thus Vertex Cover is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 1.

6. An instance of the Hamiltonian Cycle problem is a graph G. There is a solution to that instance if G has a Hamiltonian cycle, defined to be a cycle which contains every vertex of G exactly once.

   There is a $\mathcal{P}$–TIME reduction of Vertex Cover to the Hamiltonian Cycle problem, and thus Hamiltonian Cycle is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 5.

7. An instance of the Traveling Salesman Problem (TSP) is an ordered pair (G, K) where G is a weighted graph and K is a number. There is a solution to that instance if G has a Hamiltonian cycle of total weight K.

   There is a $\mathcal{P}$–TIME reduction of Hamiltonian Cycle to TSP, and thus TSP is $\mathcal{NP}$–complete.

   In my judgement, finding this reduction has difficulty level 1.