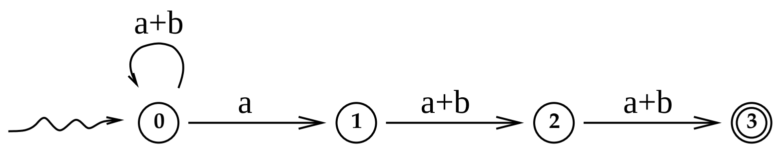
University of Nevada, Las Vegas Computer Science 456/656 Fall 2019

Answers to Assignment 3: Due Monday September 30, 2019

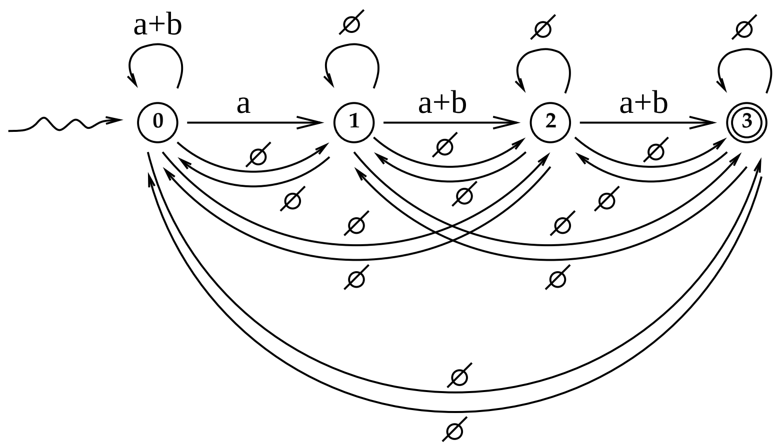
1. Write a regular expression for the language of all strings over $\{a, b\}$ which contain the substring aaa.

   $(a + b)^*aaa(a + b)^*$

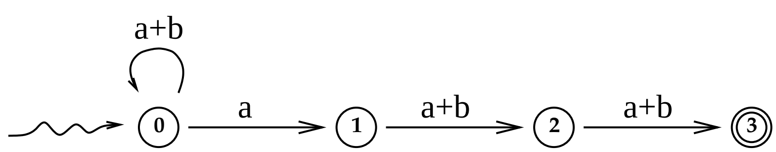
2. Use the method given on page 86 of the sixth edition of Linz, or on page 89 of the fifth edition, to find a regular expression equivalent to the following NFA.



The first step is to make it into a GTG by adding arcs between states:

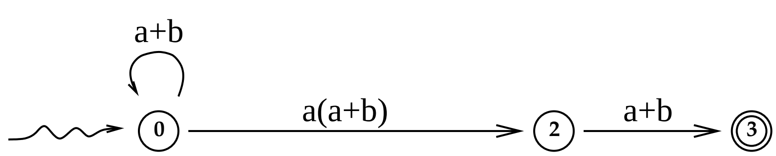


I'm sure you'll agree that leaving out arcs labeled $\emptyset$ diminishes clutter, so we'll go back to the original diagram. If any arc is missing, it is assumed to have label $\emptyset$.

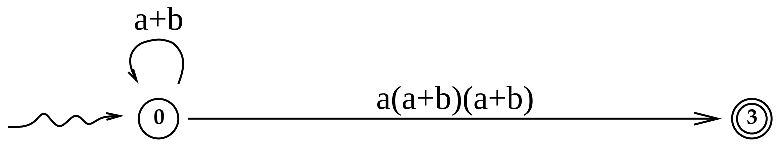


Thus $r_{00} = a + b$, $r_{01} = a$, $r_{12} = a + b$, $r_{23} = a + b$, and all other labels are $\emptyset$.

We now apply Rule 5 to eliminate state 1. The only changed label is r_{02} which becomes $a\emptyset^*(a + b) = a\lambda(a + b) = a(a + b)$.



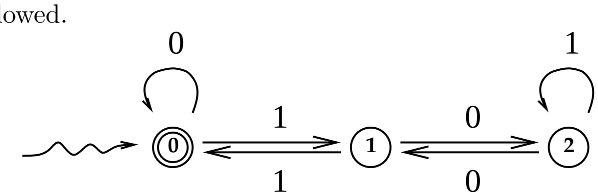
We now apply Rule 4 to eliminate state 2.



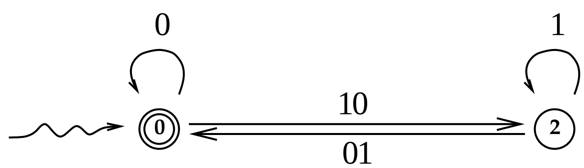
Finally, we apply Rule 3 to obtain a regular expression equivalent to the original NFA:

$$(a+b)^*a(a+b)(a+b)$$

3. The following DFA accepts the language consisting of all binary numerals for positive multiples of three, where a leading 0 is allowed.



Apply Rule 4 to eliminate state 1.



By Rule 3, the regular expression is

$$0^*10(1+010^*10)^*$$

Can that expression be simplified?

4. (a) State the pumping lemma for regular languages. We need quantifiers nested five deep.

For any regular language L,

there exists a positive integer p suchthat:

for any $w \in L$, if $|w| \geq p$,

there exist strings x, y, and z such that the following conditions hold:

i. $w = xyz$

ii. $|xy| \leq p$

iii. $|y| \geq 1$

iv. for any non-negative integer i:

$xy^iz \in L$

(b) Use the pumping lemma to prove that the language $L = \{a^n b^n : n \leq 0\}$ is not regular.

It helps so state the contrapositive of the pumping lemma, which is logically equivalent to the pumping lemma. (Note, the contrapositive of "All dogs have fur," is "If anything does not have fur, it is not a dog.") The contrapositive of the pumping lemma is,

Let L be a language such that:

for any positive integer p,

there exists a string $w \in L$, where $|w| \geq p$ such that:

If x, y, z are strings, then one of the following conditions holds:

v. $w \neq xyz$,

vi. $|xy| > p$

vii. $|y| = 0$, that is, $y = \lambda$,

viii. there exists a non-negative integer i such that:

$xy^iz \notin L$

Then L is not regular.

We now use this contrapositive to prove that $L = \{a^n b^n | n \geq 0\}$ is not regular.

Proof: Let $L = \{a^n b^n : n \geq 0\}$.

Let $p \geq 1$ be and integer.

Choose $w = a^p b^p$. (Note that $w \in L$.)

Let x, y, z be strings.

We prove, by contradiction, that one of the four conditions ix–xii holds. Suppose none of them hold. Then

ix. $w = xyz$, and

x. $|xy| \leq p$, and

xi. $|y| \geq 1$, and

xii. for any integer $i \geq 0$, $xy^iz \in L$.

By condition ix, x must be a prefix of $a^p b^p$. By condition x, xy consists entirely of a. and thus y consists entirely of a. Thus $y = a^k$. for some k. By condition xi, $k \geq 1$. Pick $i = 2$. By condition xii, $xyyz \in L$. But $xyyz = a^{p+k} b^p$. Since $k \geq 1$, $xyyz \notin L$, contradiction.

Thus, one of the conditions v–xiii holds. We can now conclude, by the contrapositive of the pumping lemma, that L is not regular. ∎

5. Work problem 9(a) on page 138 of the sixth edition, which is problem 7(a) on page 137 of the fifth edition. Find a context-free grammar for $L = \{a^n b^m : n \leq m + 3\}$.

L is generated by the following unambiguous CF grammar.

$S \to aSb \,|\, Tb \,|\, aA \,|\, \lambda$

$T \to Tb \,|\, \lambda$

$A \to aB \,|\, \lambda$

$B \to a|\lambda$

6. Work problem 9(c) on page 138 of the sixth edition, which is problem 7(c) on page 137 of the fifth edition. Find a context-free grammar for $L = \{a^n b^m \, : \, n \neq 2m\}$.

   L is generated by the following unambiguous CF grammar.

   $S \to aaSb \,|\, Tb \,|\, aA$

   $T \to Tb \,|\, \lambda$

   $A \to aA \,|\, \lambda$

7. Work problem 24 on page 140 of the sixth edition, which is problem 22 on page 139 of the fifth edition.

   (a) We give a recursive definition of "properly nested." We say that a string w over the alphabet $\Sigma = \{(,),[,]\}$ is properly nested if $w = \lambda$, or if $w = [x]y$ or $w = (x)y$, where x and y are properly nested.

   (b) Find a context-free grammar for the set of properly nested strings over Σ.

   The following unambiguous CF grammar generates the language of properly nested strings over Σ.

   $S \to (S)S$
   $S \to [S]S$
   $S \to \lambda$

8. Work problem 25 on page 140 of the sixth edition, which is problem 23 on page 139 of the fifth edition. Let L be the language of all regular expressions over $\{a, b\}$. the alphabet for L is $\Sigma = \{a, b, +, ^*, (,), \lambda, \emptyset\}$. The following CF grammar for L is ambiguous, since it doesn't enforce operator precedence. The start symbol is E (for expression).

   $E \to a \,|\, b \,|\, \lambda \,|\, \emptyset \,|\, EE \,|\, E^* \,|\, (E)$

   On the other hand, the following grammar is unambiguous, and enforces the operator precedence of regular expressions. The variables are E, for expression, T, for term, and F, for factor.

   $E \to E + T \,|\, T$

   $T \to TF \,|\, F$

   $F \to a \,|\, b \,|\, \lambda \,|\, \emptyset \,|\, F^* \,|\, (E)$