A rational number is a number that can be expressed as the ratio of two integers n and m so that n and m have no common factor. A real number that is not rational is said to be irrational. Show that $\sqrt{2}$ is irrational.

As in all proofs by contradiction, we assume the contrary of what we want to show. Here we assume that $\sqrt{2}$ is a rational number so that it can be written as

$$\sqrt{2} = \frac{n}{m}, \tag{1.5}$$

where n and m are integers without a common factor. Rearranging (1.5), we have

$$2m^2 = n^2.$$

Therefore, n^2 must be even. This implies that n is even, so that we can write $n = 2k$ or

$$2m^2 = 4k^2,$$

and

$$m^2 = 2k^2.$$

Therefore, m is even. But this contradicts our assumption that n and m have no common factors. Thus, m and n in (1.5) cannot exist and $\sqrt{2}$ is not a rational number.

This example exhibits the essence of a proof by contradiction. By making a certain assumption we are led to a contradiction of the assumption or some known fact. If all steps in our argument are logically sound, we must conclude that our initial assumption was false.

EXERCISES

1. Use induction on the size of S to show that if S is a finite set, then $|2^S| = 2^{|S|}$.
2. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

3. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1||S_2|$.
4. Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.
5. Prove DeMorgan's laws, Equations (1.2) and (1.3).

6. Occasionally, we need to use the union and intersection symbols in a manner analogous to the summation sign $\sum$. We define

$$\bigcup_{p \in \{i,j,k,\ldots\}} S_p = S_i \cup S_j \cup S_k \cdots$$

with an analogous notation for the intersection of several sets.

With this notation, the general DeMorgan's laws are written as

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

Prove these identities when P is a finite set.

7. Show that

$$S_1 \cup S_2 = \overline{\overline{S_1} \cap \overline{S_2}}.$$

8. Show that $S_1 = S_2$ if and only if

$$(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) = \varnothing.$$

9. Show that

$$S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = S_2.$$

10. Show that the distributive law

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$$

holds for sets.

11. Show that

$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3)$$

12. Show that if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$..

13. Give conditions on S_1 and S_2 necessary and sufficient to ensure that

$$S_1 = (S_1 \cup S_2) - S_2.$$