Recursive Enumeration and Canonical Order

We write $\mathcal{N}$ to mean the *natural numbers*, the numbers you learned in pre-school: the numbers people used before anyone invented negative numbers, rational numbers, real numbers, complex numbers, and even zero. In other words, the positive integers.

Every language is enumerable. We say that a language L is *recursively enumerable* if there is a machine which computes an enumeration of L. Every recursive (that is, decidable) language is recursively enumerable, but not vice-versa.

The language HALT, which we define to be $\{\langle M\rangle w : M \text{ halts with input } w\}$ is recursively enumerable (RE) but not decidable.

Theorem 1 *A language L is accepted by some machine if and only if L is recursively enumerable.*

Paradox!

Let $\mathcal{M}$ be the class of machines which take one natural number as input and give one natural number as output. Thus, each $M \in \mathcal{M}$ computes a function $\mathcal{N} \to \mathcal{N}$. Since it can be computed, that function is recursive. For each $M \in \mathcal{M}$, pick a string $\langle M\rangle$ which describes M. Let $L_{fnc} = \{\langle M\rangle : M \in \mathcal{M}\}$.

L_{fnc} is a language, hence has an enumeration. Let $\langle M_1\rangle, \langle M_2\rangle \ldots$ be an enumeration of L_{fnc}, and let $f_i : \mathcal{N} \to \mathcal{N}$ be the function computed by M_i. By our definition of L_{fnc}, every recursive (computable) $f : \mathcal{N} \to \mathcal{N}$ is equal to f_i for some i.

Now use diagonalization. We define a function $f : \mathcal{N} \to \mathcal{N}$ as follows. For any $n \in \mathcal{N}$, let $f(n) = 1 + f_n(n)$. Then f is not equal to any f_i since $f(i) = f_i(i) + 1$, hence is not recursive. But my definition gives an easy computation of f. Contradiction!

Canonical Order

Every language L has a *canonical* order, defined as follows. For $u, v \in L$, we say that u is earlier than v in canonical order if $|u| < |v|$, or if $|u| = |v|$ and u is before v in lexical (alphabetical) order.

For example, if $\Sigma = \{0, 1\}$, the binary alphabet, the canonicl order of Σ^* is

$$\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots$$

Theorem 2 *A language L is decidable if and only if there is a machine which enumerates L in canonical order.*