University of Nevada, Las Vegas Computer Science 456/656 Fall 2024

Assignment 4: Due October 12, 2024, 11:59 PM

Name:____

You are permitted to work in groups, get help from others, read books, and use the internet. You will receive a message from the graduate assistant, Sabrina Wallace, telling you how to turn in the assignment.

$\mathcal{P}$ means $\mathcal{P}$–TIME.

1. True/False. If the answer is not known to science at this time, enter "O" for Open.

   (i) **T** co-$\mathcal{P} = \mathcal{P}$.

   (ii) **O** co-$\mathcal{NP} = \mathcal{NP}$.

   (iii) **T** co-$\mathcal{P}$–SPACE = $\mathcal{P}$–SPACE.

   (iv) **T** Block placement problems are $\mathcal{NP}$.

   (v) **T** Sliding block problems are $\mathcal{P}$–SPACE.

   (vi) **O** $\mathcal{P}$–SPACE = $\mathcal{NP}$

   (vii) **O** Regular expression equivalence is $\mathcal{P}$.

   (viii) **T** Regular expression equivalence is decidable.

   (ix) **F** Context-free grammar equivalence is decidable.

   (x) **T** Every regular language is context-free.

   (xi) **F** The language C++ is context-free.

   (xii) **F** The intersection of any two context-free languages is context-free.

   (xiii) **F** The complement of any context-free language is context-free.

   (xiv) **T** Every language is countable.

   (xv) **F** For any real number x, there is a program that prints the decimal expansion of x.

   (xvi) **F** For any real number x, there is a machine that decides whether a fraction is less than x.

   (xvii) **T** There are only countably many decidable binary languages.

   (xviii) **T** Given a regular grammar G with n variables, there exists an NFA with n variables that accepts $L(G)$.

   (xix) **T** $\left\{a^ib^jc^k \,:\, i \neq j \textit{ or } j \neq k\right\}$ is a context-free language.

   (xx) **O** Given an integer n written in binary notation, it is possible to find the prime factors of n in polynomial time.

   (xxi) **T** Given an integer n written in binary notation, it is possible to decide whether n is prime in polynomial time.

   (xxii) **F** Any language generated by a grammar is decidable.

   (xxiii) **T** The complement of any decidable language is decidable.

   (xxiv) **T** The union of any two decidable languages is decidable.

(xxv) **T** The complement of any undecidable language is undecidable.

(xxvi) **F** The union of any two undecidable languages is undecidable.

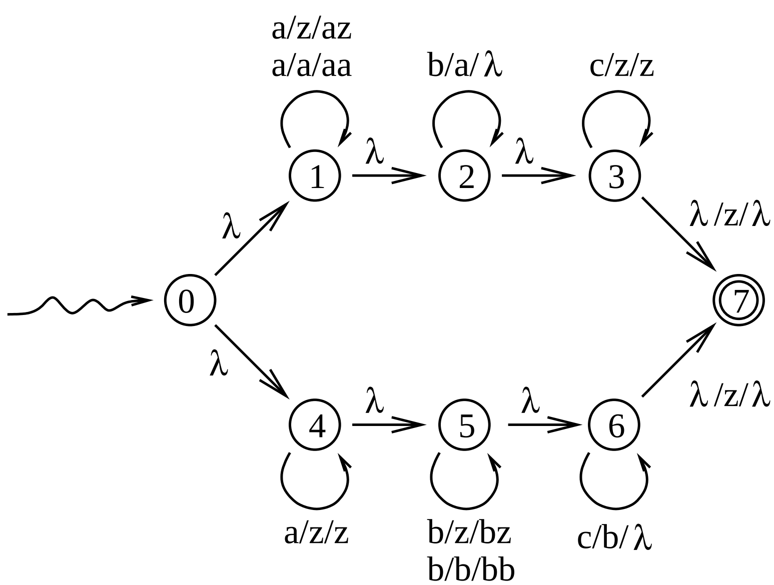
(xxvii) **F** Every context-free language is accepted by some DPDA.

(xxviii) **F** If some machine writes an increasing sequence of fractions which converges to x, then x must be a recursive real number.

2. State the pumping lemma for regular languages. If your answer contains all the right words, but not in the right order, you might get no credit.

   For any regular language L
   there exists a number p, called the pumping length of L, such that
   for any $w \in L$ of length at least p
   there exist strings x, y, z such that the following four statements hold:
   1. $w = xyz$
   2. $|xy| \leq p$
   3. $|y| \geq 1$
   4. for any integer $i \geq 0$, $xy^iz \in L$

3. Draw a PDA which accepts the language $L = \left\{a^ib^jc^k \;:\; i = j \;or\; j = k\right\}$



The PDA shown above is not a DPDA. In fact, there is no DPDA that accepts L.

4. List the names (not the definitions) of three $\mathcal{NP}$-complete problems (languages) that we have **not** discussed in class.

   There are lots of them listed on the internet.

5. Given languages L_1 and L_2, exactly one of the following statements is correct. Which one? (i)

   (i) If there is an easy reduction from L_1 to L_2 and L_1 is hard, then L_2 must be hard.

(ii) If there is an easy reduction from L_1 to L_2 and L_2 is hard, then L_1 must be hard.

(iii) If there is an easy reduction from L_1 to L_2 and L_1 is easy, then L_2 must be easy.

(iv) If there is a hard reduction from L_1 to L_2 and L_2 is easy, then L_1 must be easy.

6. Explain the verification definition of the class $\mathcal{NP}$. Do not write more than necessary. Your answer should be concise and correct.

   A language L is $\mathcal{NP}$ if and only if there exist a machine M and an integer k such that

   (i) For any $w \in L$, there exists a string c (called a *cerficiate* for w) such that M accepts the string w, c in $O(n^k)$ time, where $n = |w|$.

   (ii) For any $w \notin L$, and for any string c, M does not accept w, c.

7. Prove that, for any positive integer n, the sum of the first n cubes, $1^3 + 2^3 + 3^3 + \cdots n^3$, is $\dfrac{n^2(n+1)^2}{4}$

   Let $F(n)$ be the sum of the first n cubes. Let $G(n) = \dfrac{n^2(n+1)^2}{4}$. We prove, by induction, that $F(n) = G(n)$ for any positive integer n.

   $G(1) = \dfrac{1^2 \cdot 2^2}{4} = 1 = F(1)$. The inductive step is that $F(n) = G(n)$ implies $F(n+1) = G(n+1)$ for any n. Suppose $F(n) = G(n)$. By definition, $F(n+1) = F(n) + (n+1)^3$. Then
   $G(n+1) = \dfrac{(n+1)^2(n+2)^2}{4} = \dfrac{(n^2+4n+4)(n+1)^2}{4} = \dfrac{n^2(n+1)^2}{4} + \dfrac{(4n+4)(n+1)^2}{4} =$
   $F(n) + (n+1)^3 = F(n+1)$

8. Prove that $\sqrt{3}$ is irrational. *Proof:* Suppose $\sqrt{3} = \frac{p}{q}$, for integers p, q. Without loss of generality, the greatest common divisor of p and q is 1. Then

$$
\begin{aligned}
\sqrt{3} &= \frac{p}{q} \\
3 &= \frac{p^2}{q^2} \\
3q^2 &= p^2
\end{aligned}
$$

   Thus p^3 is a multiple of 3, and hence p is a multiple of 3, that is, $p = 3k$ for some integer k.

$$
\begin{aligned}
3q^2 &= 9k^2 \\
q^2 &= 3p^2
\end{aligned}
$$

   Thus q^2 is a multiple of 3, and hence q is a multiple of 3, which contradicts the fact that p and q have no common divisor greater than 1.

   We conclude that $\sqrt{3}$ is irrational. ▮

9. Prove that $\log_2 3$ is irrational. (Hint: What is the definition of logarithm?)

   Proof: Assume $\log_2 3$ is rational, hence equal to $\frac{p}{q}$ for positive integers p and q. By definition of the logarithm, we have $3 = 2^{\log_2 3} = 2^{\frac{p}{q}}$. Take both sides to the power of q, we have $3^q = 2^p$. But any power of 3 must be odd and any positive power of 2 must be even, contradiction. Thus, $\log_2 3$ is irrational. ∎

10. However, $\log_2 3$ is very close to the rational number 19/12, only about 1% off. Explain why this fact is important for Western music.[1]

    A combination of notes, the ratios of whose frequencies are simple fractions is pleasing to the ear. Playing C and G together sounds pleasing, since the frequency of G is ideally $3/2$ times the frequency of C. For every note on the piano, there should be a note whose frequency is $3/2$ times as much. If you play C, then G above that, then D above that, and so forth until you reach C, the frequency should be increased by a factor of $(3/2)^{12}$, and you are exactly 7 octaves higher, which means the frequency is incrased by a factor of 2^7, which is not exactly $(3/2)^{12}$. You could redesign the keyboard to get it closer, but you can never get it perfect, since $\log_2 3$ is irrational. One solution is the even tempered piano, where the frequcies of keys are evenly spaced logarithmically, that is, the frequency ratio of consecutve keys is $\sqrt[12]{2}$. In this system, the frequency ratio from C to G is $(\sqrt[12]{2})^7$, which is approximately 1.498307077, close to 3/2.

[1] From the internet: "Western music may be defined as organized instrumentation and sound created and produced in Europe, the United States, and other societies established and shaped by European immigrants. This includes a wide assortment of musical genres, from classical music and jazz to rock and roll and country-western music."