# The Halting Problem is Undecidable

We define the language HALT to be the set of all strings of the form $\langle M \rangle w$ such that $M$ halts with input $w$. HALT is the language which is equivalent to the halting problem.

**Theorem 1** HALT *is not decidable.*

*Proof:* By contradiction. Suppose HALT is decidable. Let $D$ be a machine which implements the following program:

read a machine description $\langle M \rangle$.<br>
if $M$ halts with input $\langle M \rangle$<br>
  run forever.<br>
else<br>
  halt.

We now run $D$ with input $\langle D \rangle$. One of the following two cases must hold.

Case 1. $D$ halts with input $\langle D \rangle$. That means that, when $D$ reads $\langle D \rangle$, it runs forever, hence $D$ does not halt with input $\langle D \rangle$, contradiction.

Case 2. $D$ does not halt with inpu $\langle D \rangle$. That means that, when $D$ reads $\langle D \rangle$, it halts, hence $D$ halts with input $\langle D \rangle$, contradiction.

In either case, we obtain a contradiction, hence HALT is undecidable. ∎

**Theorem 2** *HALT is recognizable.*

*Proof:* The following program $P$ recognizes HALT.

read $\langle M \rangle w$<br>
run $M$ with input $w$.<br>
if $M$ halts with input $w$<br>
  accept $\langle M \rangle w$.

Thus, $P$ accepts every member of HALT, but no other string. ∎

Note that the program will run forever if $\langle M \rangle w \notin$ HALT. HALT is recursively enumerable, since every language recognized by a machine is recursively enumerable.

**Theorem 3** *Every recursively enumerable language is decidable.*

*Proof:* Let $L$ be a language enumerated by a machine $M$. Let $w_1, w_2, \ldots$ be the enumeration of $L$ given by $M$. For any $n \geq 0$, let $L[n]$ be the finite set of all members of $L$ of length no greater than $n$. Each member of $L[n]$ must be $w_i$ for some $i$. Define $T(n) = max(i : w_i \in L[n])$. The following program decides $L$.

read $w$, let $n = |w|$<br>
If($M$ writes $w$ within time $T(n+1)$), ACCEPT $w$. ∎<br>
Else REJECT $w$.

Thus, HALT is decidable. But it's undecidable! What's wrong with this proof?