CSC 456/656 Fall 2024 Answers to Second Examination October 23, 2024

The entire exam is 270 points.

In the questions of this test, $\mathcal{P}$ and $\mathcal{NP}$ denote $\mathcal{P}$-TIME and $\mathcal{NP}$-TIME, respectively.
If L is a language over an alphabet Σ, we define the *complement* of L to be the set of all strings over Σ which are not in L. If $\mathcal{C}$ is a class of languages, we define co-$\mathcal{C}$ to be the class of all complements of members of $\mathcal{C}$.

1. True or False. 5 points each. T = true, F = false, and O = open, meaning that the answer is not known science at this time.

   (i) **F** The complement of a CFL is always a CFL.

   (ii) **F** The class of context-free languages is closed under intersection.

   (iii) **T** The language of binary numerals for multiples of 23 is regular.

   (iv) **T** The set of binary numerals for prime numbers is in $\mathcal{P}$–TIME.

   (v) **O** The complement of any $\mathcal{NP}$ language is $\mathcal{NP}$.

   (vi) **T** The complement of any undecidable language is undecidable.

   (vii) **O** $\mathcal{NP} = \mathcal{P}$–SPACE.

   (viii) **T** The complement of any $\mathcal{P}$–TIME language is $\mathcal{P}$-TIME.

   (ix) **T** The complement of any $\mathcal{P}$–SPACE language is $\mathcal{P}$–SPACE.

   (x) **F** The complement of every recursively enumerable language is recursively enumerable.

   (xi) **T** π is recursive.

   (xii) **T** If $\mathcal{P} = \mathcal{NP}$, then all one-way encoding systems are breakable in polynomial time.

   (xiii) **T** A language L is in $\mathcal{NP}$ if and only if there is a polynomial time reduction of L to SAT.

   (xiv) **T** If L_1 reduces to L_2 in polynomial time, and if L_2 is $\mathcal{NP}$ and L_1 is $\mathcal{NP}$-complete, then L_2 is $\mathcal{NP}$-complete.

   (xv) **O** The question of whether two regular expressions are equivalent is $\mathcal{NP}$-complete.

   (xvi) **T** Every language which is accepted by some non-deterministic machine is accepted by some deterministic machine.

   (xvii) **F** The language of all regular expressions over $\{a, b\}$ is a regular language.

(xviii) **F** The equivalence problem for C++ programs is recursive.

(xix) **F** Every function that can be mathematically defined is recursive.

(xx) This problem is a duplicate.

(xxi) **T** If there is a recursive reduction of the halting problem to a language L, then L is undecidable.

(xxii) **F** If there is a recursive reduction of a language L to the halting problem, then L is undecidable.

(xxiii) **T** The set of rational numbers is countable.

(xxiv) **F** The set of real numbers is countable.

(xxv) **T** The set of recursive real numbers is countable.

(xxvi) **T** There are countably many recursive binary functions.

(xxvii) **T** The context-free grammar equivalence problem is co-$\mathcal{RE}$.

(xxviii) **T** Let $L = \{\langle G_1 \rangle, \langle G_2 \rangle \, : \, L(G_1) \neq L(G_2)\}$ Then L is recursively enumerable.

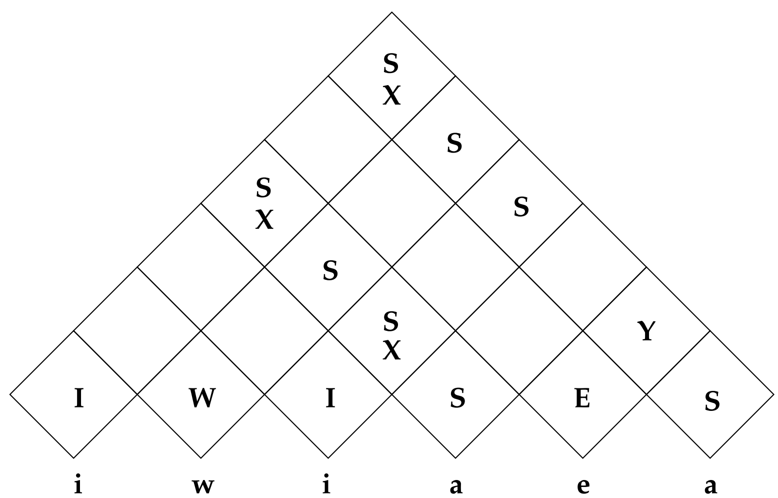
(xxix) **T** The factoring problem for unary numerals is $\mathcal{P}$–TIME

(xxx) **T** co-$\mathcal{P}$–TIME $=$ $\mathcal{P}$–TIME

(xxxi) **T** If L is an $\mathcal{RE}$ (recursively enumerable) language, there is a program that accepts L.

2. [20 points] Let L be the language generated by the Chomsky Normal Form (CNF) grammar given below. Use the CYK algorithm to prove that the string *iwiaea* is a member of L. Use the figure below for your work.

$S \to IS$
$S \to XY$
$S \to WS$
$S \to a$
$X \to IS$
$Y \to ES$
$W \to w$
$I \to i$
$E \to e$



3. [20 points] State the pumping lemma for regular languages.

If L is a regular language
There exists a positive number p such that
For any $w \in L$ of length at least p
There exist string x, y, z such that the following hold

1. $w = xyz$
2. $|xy| \leq p$
3. $|y| \geq 1$
4. For any integer $i \geq 0$ $xy^iz \in L$

4. [20 points] Let $L = \{w \in \{a,b\}^* : \#_a(w) = \#_b(w)\}$, all strings over $\{a,b\}$ which have equal numbers of a's and b's. Use the pumping lemma to prove that L is not regular.

Theorem 1 *L is not regular.*

Proof: By contradiction. Assume L is regular. Let p be the pumping length of L. Let $w = a^pb^p \in L$, which has length greater than p. By the pumping lemma, there exist strings x, y, z such that

1. $w = xyz$
2. $|xy| \leq p$
3. y is not the empty string
4. For any $i \geq 0$ $xy^iz \in L$

By 1. xy is a prefix of w. By 2., xy is a substring of a^p. By 3. $y = a^k$ for some $k > 0$, hence $xy^0z = xz = a^{p-k}b^k \in L$, which is a contradiction to the requirement that each string in L has equally many of each symbol. We conclude that L is not regular. ∎

5. [20 points] Give a polynomial time reduction of the subset sum problem to partition.

Let $X = (x_1, x_2, \ldots x_n, K)$ be an instance of the subset sum problem. Let $S = \sum_{i=1}^{n} x_i$. We assume $K \leq X$, since otherwise the instance has no solution. Let $Y = (x_1, x_2, \ldots x_n, K+1, S-K+1)$, an instance of the partition problem. Then Y has a solution if and only if X has a solution.

6. [20 points] Give a polynomial time reduction of 3-SAT to the Independent Set problem.

Let $E = C_1 \cdot C_2 \cdot \cdots \cdot C_k$ be a Boolean expression in 3-CNF form. For each i, let C_i be the disjunction of three terms, $t_{i,1} + t_{i,2} + t_{i,3}$ where each $t_{i,j}$ is either a variable or the negation of a variable. Let G be the graph with $3k$ vertices, $\{v_{i,j}\}$ each correponding to a term of E. We let G have an edge from $v_{i,j}$ to $v_{p,q}$ if either $i = p$ or the expression $t_{i,j} \cdot t_{p,q}$ is a contradiction, that is, one of those terms is a variable and the other the negation of that variable. Then E has a satisfying assignment if and only if G has an independent set of size k.

7. [20 points] Prove that $\sqrt{2}$ is irrational.

Theorem 2 *$\sqrt{2}$ is irrational.*

Proof: By contradiction. Assume $\sqrt{2}$ is rational. Then $\sqrt{2}$ is can be written as a fraction $\frac{p}{q}$, where p and q are integers with no common divisor greater than 1. Then

$\sqrt{2} = \frac{p}{q}$

$2 = \frac{p^2}{q^2}$

$2q^2 = p^2$

Hence p^2 is even

Hence p is even.

Write $p = 2k$ where k is an integer

$p^2 = 4k^2$

$2q^2 = 4k^2$

$q^2 = 2k^2$

Hence q^2 is even

Hence q is even.

p and q have a common divisor 2

Contradiction. Therefore, $\sqrt{2}$ is not rational. ∎