Real Numbers

There are uncountably many real numbers. However, only countably many of these can be "known" in some sense.

Recursive Real Numbers

There are several equivalent definitions of recursive real numbers. Here are three of them.

(a) A real number is recursive if some machine can enumerate its decimal (or binary, or whatever) expansion.

(b) A real number x is recursive if there is a Boolean valued recursive function $less[x](p, q)$ which is true if and only if the fraction p/q is less than x.

(c) A real number x is recursive if there is a recursive function digit[x](n) which returns the n^{th} digit of the decimal (or whatever) expansion of x.

There are uncountably many real numbers, but only countably many of these are recursive.

Definable Real Numbers

We say that a real number x is definable if there is some Boolean-valued expression with one real parameter such that x satifies the expression and no other real number satisfies the expression. Here is an example: "$x^2 = 2$ and $x > 0$" In this case, $x = \sqrt{2}$, which is also recursive. All recursive real numbers are definable, but not all definable real numbers are recursive.

There are only countably many definable real numbers, since there are only countably many expressions.

A Real-valued Function on Languages. Let Σ be an alphabet, and let $w_1, w_2, \ldots$ be the canonical ordering of Σ^*. For any language L over Σ define

$$f(L) = \sum_{i=1}^{\infty} \lambda_i 2^{-i} \text{ where } \lambda_i = \begin{cases} 1 \text{ if } w_i \in L \\ 0 \text{ otherwise} \end{cases}$$

Each of the following statements is true regardless of the choice of Σ. If it makes you more comfortable, you may assume that Σ is the binary alphabet $\{0, 1\}$.

Properties of the function f:

1. For any language L over Σ, $0 \leq f(L) \leq 1$.
2. If $L_1 \subseteq L_2$ then $f(L_1) \leq f(L_2)$.
3. If L' is the complement of L, then $f(L) + f(L') = 1$.
4. $f(L)$ is recursive if and only if L is decidable.
5. $f(L)$ is definable if L is recursively enumerable.

   (But not "if and only if." The converse of the statement is false.)
6. For every real number x in the range $0 \leq x \leq 1$, there is some language L over Σ such that $f(L) = x$.

A Paradox. According to Property 6 every x in the interval is $f(L)$ for some language L. The definition of $f(L)$ then gives us a mathematical definition of x, so x is definable. There are uncountably many numbers in the interval between 0 and 1, but there are only countably many definable real numbers. Contradiction!