Closure under Kleene Star for Certain Classes

If L is a language, we let L^* be the Kleene star of L, also called the Kleene closure of L.

The Substring Graph

Let w be any string, and let $n = |w|$. We define the *substring graph* of w to be the labeled acyclic directed graph $S(w)$ with vertices $0, 1, \ldots n$ and arcs (i, j) for $0 \le i < j \le n$. Each arc (i, j), with $i < j$, is labeled with the non-empty substring $w_{i+1,j}$. $S(w)$ has source 0 and sink n

Fix a language L. Let L' denote the complement of a language L. We define the L-coloration of $S(w)$. The edge (i, j) is colored black if $w_{i+1,j} \in L$, otherwise red.

Remark 1 *The L-colored substring graph of w contains a black path from the source to the sink if and only if $w \in L^*$, and contains a red path from the source to the sink if and only if $w \in (L')^*$.*

$\mathcal{RE}$ and co-$\mathcal{RE}$

A language is in the class $\mathcal{RE}$ if and only if there is a deterministic machine M which accepts L, *i.e.* accepts every string $w \in L$ and does not accept any string not in L. A language L is in the class co-$\mathcal{RE}$ if and only if its complement is in the class $\mathcal{RE}$, *i.e.* there is a deterministic machine M which accepts every string $w \notin L$ and accepts no string in L. Consider the following two programs:

Program $P_1(w)$
Color every edge of the substring graph $G(w)$ red.
Emulate M simultaneously on all substrings of w. Each emulation runs forever, or until the emulation halts or the program halts.
At any step, if the emulation halts for one substring, color the corresponding arc of the substring tree black.
If there is a black path in $G(w)$ from the source to the sink, halt and accept.

Program $P_2(w)$
Color every edge of the substring graph $G(w)$ red.
Emulate M simultaneously on all substrings of w. Each emulation runs forever, or until the emulation halts or the program halts.
At any step, if the emulation halts for one substring, color the corresponding arc of the substring tree black.
If there is no red path in $G(w)$ from the source to the sink, halt and accept.

Lemma 1 *Program P_1 accepts L^*.*

Lemma 2 *Program P_2 accepts $((L')^*)'$.*

Proof: If either program runs forever, the coloring on $G(w)$ converges to the L-coloring. Recall that $w \in L^*$ if and only if there is a black path from 0 to n in the L-coloring, while $w \in (L')^*$ if and only if there is a red path from 0 to n in the L-coloring. Since no arc changes from black to red, once there is a black path from the source to the sink, there will always be such a path, hence $w \in L^*$ This proves Lemma 1. Similarly, once there is no red path from source to the sink, there will never again be such a path, hence $w \notin (L')^*$. This proves Lemma 2. ∎

From Lemma 1, we have that $\mathcal{RE}$ is closed under Kleene star, while from Lemma 2, we have that co-$\mathcal{RE}$ is closed under Kleene star.

Unfinished