Closure

Introduction

A set, or class, is *closed* under an operation if, whenever that operation is applied to members of that class, the resulting object is a member of that class. For example, the set of integers is closed under the operations addition, subtraction, multiplication, additive inverse, and exponentiation, but not under multiplicative inverse or division.

Other Number Systems

The set of real numbers is closed under addition, subtraction, multiplication, addititive inverse, but not under mulitiplicative inverse, division, or exponentiation (such as x^y). Why?
Ans:
0 has no multiplicative inverse, you can't divide by zero, and you can't take a negative number to a fractional power, like $(-1)^{\frac{1}{2}}$.

Which of the following is the set of positive real numbers closed under? (T/F)

a) Addition ________

b) Subtraction ________

c) Multiplication ________

d) Division ________

e) Addititive inverse ________

f) Multiplicative inverse ________

g) Square root ________

h) Exponentiation ________

i) Base 2 logarithm ________

0.0.1 Language Classes

A language is a set, therefore set operations apply. We use either $\cup$ or $+$ to denote union, and $\cap$ to denote intersection. If L is a language over an alphabet Σ, the complement of L is defined to be all strings over Σ not in L.

The *concatenation* of two strings u and v, is written uv. For example, if $u = ab$ and $v = bab$, then $uv = abbab$.

If L_1, L_2 are languages, the concatenation is defined as $L_1L_2 = \{uv : u \in L_1 \text{ and } v \in L_2\}$ If $L_1 = \{a, ab\}$ and $L_2 = \{\lambda, b\}$ then $L_1L_2 = \{a, ab, abb\}$. Concatenation is associative, thus $L_1L_2L_3$ is defined.

We use exponentiation to indicated repeated concatenation. That is, $L^2 = LL$, $L^3 = LLL$, and so forth. We let $L^1 = L$ and $L^0 = \{\lambda\}$.

If L is a language, then $L^* = L^0 + L^+ L^2 + L^3 + \cdots = \sum_{i \geq 0} L^i$ is the *Kleene closure* of L. For example, if $L = \{a, aaa\}$, then $L^* = \{\lambda, a, aa, aaa, aaaa, \ldots\} = \{a^n : n \geq 0\}$

What is the Kleene closure of the empty language?

In our course, we sometimes define a language class using grammars, and sometimes using computational hardness.

1. The class of regular languages is closed under
   (a) union
   (b) intersection
   (c) complementation
   (d) concatenation
   (e) Kleene closure
2. The class of context-free language is closed under
   (a) union
   (b) concatenation
   (c) Kleene closure

   but not intersection or complementation.
3. The class of polynomial-time languages $\mathcal{P}$-TIME, sometimes known simply as $\mathcal{P}$, is defined to be all languages which are decided by some machine in polynomial time. $\mathcal{P}$–TIME is closed under
   (a) union
   (b) intersection
   (c) concatenation
   (d) Kleene closure
   (e) complementation
4. $\mathcal{NP}$–TIME, or simply $\mathcal{NP}$, is the class of all languages which are accepted by some non-deterministic machine in polynomial time. $\mathcal{NP}$–TIME is closed under
   (a) union
   (b) intersection
   (c) concatenation
   (d) Kleene closure

   Is $\mathcal{NP}$–TIME closed under complementation?