Pumping Lemmas

The main usefulness of the two pumping lemmas is to prove that a particular language is not regular, or context-free, as the case may be. Each lemma states that every language in the class has a certain property, and thus if we can prove that a given language L does not have that property, L is not in the class.

Lemma 1 (Pumping Lemma for Regular Languages) *If L is a regular language, there exists a positive integer p, called the pumping length of L, such that for any string $w \in L$ whose length is at least p, there exist strings x, y, z such that the following conditions hold.*

1. $w = xyz$
2. $|y| \geq 1$
3. $|xy| \leq p$
4. *for any* $i \geq 0$, $xy^i z \in L$.

Note that the the value of p is not unique: if p is a pumping length of L, so is every integer larger than p. There is a minimum pumping length.

Example

Let L be the language of all base 2 numerals for multiples of 5, where leading zeros are not allowed. The minimum pumping length is 5. We won't prove that, but for example, if $w = 11001$, which means 25, we let $x = 1$, $y = 10$, and $z = 01$. The first three conditions obviously hold. If we let $i = 0$, we get $xz = 101$, which means 5, while if $i = 2$ or $i = 3$, we get $xy^2z = 1101001$ which means 105, or $xy^3z = 110101001$ which means 425. The pumping length cannot be 4, since 1111, which means 15, does not have a pumpable substring. Thus, 5 is minimum.

Another example is $w = 1110011$, which means 115. Let $x = 11$, $y = 100$, and $z = 11$.

Lemma 2 (Pumping Lemma for Context-Free Languages) *If L is a context-free language, there exists a positive integer p, called the pumping length of L, such that for any string $w \in L$ whose length is at least p, there exist strings u, v, x, y, z such that the following conditions hold.*

1. $w = uvxyz$
2. $|v| + |y| \geq 1$
3. $|vxy| \leq p$
4. *for any* $i \geq 0$, $uv^ixy^iz \in L$.

Note that the the value of p is not unique: if p is a pumping length of L, so is every integer larger than p. There is a minimum pumping length.

Example

Let L be the language consisting of all palindromes over $\{a, b\}$. The following is an unambiguous grammar for L.

$S- > aSa|bSb|a|b|\lambda$

What is the minimum pumping length of L?

The answer is 3. If a palindrome w has even length, the substring aa or bb in the middle of the string. That is, $w = uaau^R$ or $w = ubbu^R$. Suppose $w = uaau^R$. We let $u = u$, $v = a$, $x = \lambda$, $y = a$, and $z = u^{\mathrm{R}}$. The first three conditions are obviously satisfied. For any $i \geq 0$, $uv^ixy^iz = ua^ia^iu^R \in L$. The case that $w = ubbu^R$ is similar.

If w has odd length, then there are four possibilities:
$w = uaaau^R$
$w = uabau^R$
$w = ubabu^R$
$w = ubbbu^R$
In the first case, we let $u = u$, $x = a$, $y = a$, and $z = u^R$. In the second case, we let $u = u$, $x = a$, $y = b$, and $z = u^R$. The four conditions are satisfied. The other two cases are similar.

The minimum pumping length cannot be 2, because $w = aba \in L$, and the four conditions cannot be fulfilled for w with $p = 2$.