Countability and Real Numbers

Different Infinities

If S is any set the set of all subsets of S is written 2^S. Georg Cantor (1845–1918) was the first to note that infinite sets may be of different sizes (cardinalities). We say that two sets A and B have the same *cardinality* if there is a 1-1 correspondents $f : A \to B$. That is, for every $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$. If S is any set, the cardinality S is written as $|S|$, and the cardinality of 2^S is written $2^{|S|}$.

The finite cardinals are called zero, one, two, *etc.* But some sets are infinite, such as $\mathcal{N}$, the set of natural numbers. We use the Hebrew letter $\aleph$ to denote infinite cardinals. By definition, $\aleph_0$ is the smallest infinite cardinal, $\aleph_1$ is the next smallest, and so forth.

The cardinality of the set of natural numbers $\mathcal{N}$ is α_0. Thus the cardinality of every enumerable (countable) set is also $\aleph_0$. The set of all integers and the set of all fractions are both countable. Every language is countable, and every subset of a countable set is countable; that subset is either finite or has cardinality $\aleph_0$.

Cantor proved, using diagonalization, that the set of real numbers $\mathbb{R}$ is uncountable. (Sets which are not countable are called uncountable.) Here is his proof, which is by contradiction.

Suppose $\mathbb{R}$ is countable. Then there is a 1-1 correspondence $f : \mathcal{N} \to \mathbb{R}$. Each real number has a decimal expansion. Let x be a real number between 0 and 1 such that the ith decimal digit of x (that is, in the 10^{-i} place) is different from the i^{th} decimal digit of $f(i)$. Then $x \neq f(i)$ for all i, and thus the image of f does not contain all real numbers, contradiction.

Another example of an uncountable set is the set of all languages over a given alphabet Σ. That set is the set of all subsets of Σ^*, and is written 2^{Σ^*}. The cardinality of that set is $2^{\aleph_0}$, which is the same as the cardinality of $\mathbb{R}$.

The continuum hypothesis is that the cardinality of $\mathbb{R}$ is $\aleph_1$, that is, $2^{\aleph_0} = \aleph_1$ Cantor died without being able to prove it. In 1964, Paul Cohen proved that the continuum hypothesis cannot be proved using the standard axioms of set theory. It was already known that it could not be disproved, either. (The room was packed when I saw Cohen's presentation.)

Recursive Real Numbers

We say that a real number x is *recursive* if one of these equivalent conditions holds.

1. There is a machine that runs forever, printing the decimal (or binary, or whatever) expansion of x.
2. The problem of whether a given fraction is less than x is decidable.

We will restrict our attention to binary expansions. I got the first few places of the binary expansion of π from the internet:

$$\pi = 110010010000111111011$$

If L is a language over the binary alphabet $\Sigma = \{0, 1\}$ define x_L to be the real number $\sum_{w_i \in L} 2^{-i}$