University of Nevada, Las Vegas Computer Science 456/656 Spring 2023

Answers to Assignment 7: Due Saturday April 29, 2023, 23:59

Name:____

You are permitted to work in groups, get help from others, read books, and use the internet.

1. Determine whether each of these 2CNF expressions is satisfiable. If satisfieable, given a satisfying assignment. Otherwise, prove the expression is a contradiction.

   (a) $(!e+!f)*(!f+!b)*(!d+g)*(e+!j)*(!e+!i)*(!e+!b)*(!f+i)*(!d+g)*(!d+f)*(f+a)$
   $*(h+i)*(!j+f)*(!d+!h)*(!c+e)*(!c+a)*(!i+!h)*(!b+e)*(a+g)*(!c+!b)*(!f+g)$

   Satisfiable. There are serveral satisfying assignments. Here is one.

   $a=1, b=0, c=0, d=0, e=0, f=1, g=1, h=0, i=1, j=0$

   (b) $(!i+f)*(h+!b)*(!h+!d)*(d+b)*(i+!i)*(e+!b)*(i+d)*(g+!d)*(!i+f)*(!f+!c)$
   $*(!c+!d)*(!b+i)*(h+i)*(!f+!h)*(!d+c)*(a+!h)*(i+d)*(!f+!a)*(!c+!h)*(c+!g)$

   Not satisfiable. Here is proof. Suppose the expression is satisfiable; Pick a satisfying assignment and assign each variable the value given by that assignment. Then each of the following clauses is true.
   $C_1 = (!d + c)$
   $C_2 = (!c+!d)$
   $C_3 = (d + b)$
   $C_4 = (!b + i)$
   $C_5 = (!b + h)$
   $C_6 = (!i + f)$
   $C_7 = (!f+!h)$

   We make repeated use of the fact that $x * (!x + y)$ implies y, for any expressions
   By C_1 and C_2, if $d = 1$, c and $!c$ must both be true, contradiction; thus $d = 0$.
   We have $!d * C_3 =!d * (d + b)$, which implies $b = 1$.
   We have $b * C_4 = b * (!b + i)$, which implies $i = 1$
   We have $b * C_5 = b * (!b + h)$, which implies $h = 1$.
   We have $i * C_6 = i * (!i + f)$, which implies $f = 1$.
   We have $f * C_7 = f * (!f+!h)$, which implies $h = 0$.
   Contradiction. Thus the original expression is a contradiction, *i.e.* not satisfiable.

2. Give a polynomial time reduction of the subset sum problem to the partition problem.

   Let $(\{x_1, x_2, \ldots, x_n\}, K)$ Be an instance of the subset sum problem. In the proof given in the handout, I assumed that all the x_i are positive. But this restriction is unnecessary.

   Pick a number N which is larger than $\sum_{i=1}^{n} |x_i|$. Let $S = \sum_{i=1}^{n} x_i$. The reduction maps that instance of the subset sum problem to the instance $\{x_1, x_2, \ldots x_n, K + N, S - K + N\}$ of the partition problem.

3. Give a proof that a recursively enumerable language is accepted by some machine.

   Let M_1 be a machine which enumerates a language L. Let $w_1, w_2, \ldots$ be that enumeration of L. Let M_2 be the machine whose computation is defined by the following program.

   Read w
   For i from 1 to ∞
     If $(w_i = w)$
       HALT and ACCEPT

   M_2 accepts L.

4. Give a proof that a language accepted by a machine is recursively enumerable.

   Let M_2 be a machine which accepts a language L over an alphabet $Sigma$. Let $\Sigma^* = \{w_1, w_2, \ldots\}$ in canonical order. Let M_1 be the machine whose computation is defined by the following program.

   For t from 1 to ∞
     For i from 1 to t
       If $(M_1$ accepts w_i within t steps)
         Write w_i

   Then M_1 enumerates L. (Repetition is allowed. Each string of L will be written infinitely many times. How would you eliminate this repetition?)

5. Give a context-sensitive grammar for $\{a^n b^n a^n \,:\, n \geq 1\}$.

   There are many correct answers. Here is one:
   $S \to aba \,|\, aaAba$
   $aA \to aaAA$
   $Ab \to bA$
   $bAa \to bbaa$
   Here is a derivation of $a^3b^3a^3$:
   $S \Rightarrow aaAba \Rightarrow aaaAAba \Rightarrow aaaAbAc \Rightarrow aaaAbbaa \Rightarrow aaabAbaa \Rightarrow aaabbAaa \Rightarrow aaabbbaaa$

   Here is a derivation of $a^4b^4a^4$:
   $S \Rightarrow aaAba \Rightarrow aaaAAba \Rightarrow aaaAbAa \Rightarrow aaaAbbaa \Rightarrow aaaaAAbbaa \Rightarrow aaaaAbAbaa \Rightarrow aaaaAbbAaa \Rightarrow aaaaAbbbaaa \Rightarrow aaaabAbbaaa \Rightarrow aaaabbAbaaa \Rightarrow aaaabbbAaaa \Rightarrow aaaabbbbaaaa$

6. Let L be the language consisting of all strings of a's of length a power of 2. That is,
   $L = \{a, aa, aaaa, aaaaaaaa, aaaaaaaaaaaaaaaa, \ldots\}$.

   (a) Give a context-sensitive grammar for L.

   Here is one:
   1. $S \to a$
   2. AC
   3. $A \to AaB$
   4. $Ba \to aaB$
   5. $BC \to aC$
   6. $C \to a$

7. $A \rightarrow a$

(b) Using the grammar you gave for 6a, give derivations of the strings a, aa, $aaaa$, and $aaaaaaaa$.

$S \stackrel{1}{\Rightarrow} a$

$S \stackrel{2}{\Rightarrow} AC \stackrel{6}{\Rightarrow} Aa \stackrel{7}{\Rightarrow} aa$

$S \stackrel{2}{\Rightarrow} AC \stackrel{3}{\Rightarrow} AaBC \stackrel{5}{\Rightarrow} AaaC \stackrel{6}{\Rightarrow} Aaaa \stackrel{7}{\Rightarrow} aaaa$

$S \stackrel{2}{\Rightarrow} AC \stackrel{3}{\Rightarrow} AaBC \stackrel{5}{\Rightarrow} AaaC \stackrel{3}{\Rightarrow} AaBaaC \stackrel{4}{\Rightarrow} AaaaBaC$
$\stackrel{4}{\Rightarrow} AaaaaaBC \stackrel{5}{\Rightarrow} AaaaaaaC \stackrel{6}{\Rightarrow} Aaaaaaaa \stackrel{7}{\Rightarrow} aaaaaaaa$

$S \stackrel{2}{\Rightarrow} AC \stackrel{3}{\Rightarrow} AaBC \stackrel{5}{\Rightarrow} AaaC \stackrel{3}{\Rightarrow} AaBaaC \stackrel{4}{\Rightarrow} AaaaBaC \stackrel{4}{\Rightarrow} AaaaaaBC \stackrel{5}{\Rightarrow} AaaaaaaC \stackrel{3}{\Rightarrow} AaBaaaaaaC \stackrel{4}{\Rightarrow} AaaaBaaaaaC \stackrel{4}{\Rightarrow} AaaaaaBaaaaC \stackrel{4}{\Rightarrow} AaaaaaaaBaaaC \stackrel{4}{\Rightarrow} AaaaaaaaaaBaaC \stackrel{4}{\Rightarrow} AaaaaaaaaaaaBaC \stackrel{4}{\Rightarrow} AaaaaaaaaaaaaaBC \stackrel{5}{\Rightarrow} AaaaaaaaaaaaaaaC \stackrel{6}{\Rightarrow} Aaaaaaaaaaaaaaaa \stackrel{7}{\Rightarrow} aaaaaaaaaaaaaaaa$

7. Prove that every context-sensitive language is recursive. (You may want to search the internet.)

   Let G be a context-free grammar which generates the language L. Let Σ and Γ be the terminal alphabet and the variable alphabet of G, respectively.

   The problem is to decide whether a given string $w \in \Sigma^*$ is in L. If $w = \lambda$, we can answer the question in $O(1)$ time. Henceforth, assume that $|w| = n > 0$.

   Recall that a *sentential form* of G is any string over $\Sigma + \Gamma$ which could occur in a derivation of G. Let $D_[k]$ be the set of sentential forms of G of length k. That set is finite. Since G is non-decreasing, $D[1] \subseteq \Sigma + \Gamma$ is easily computed, and $D[k]$ can be computed from $D[k-1]$. Then $w \in L$ if and only if $w \in D[n]$.