

General Grammars

The following grammar generates $L = \{a^{n^2} : n \geq 0\}$. The only terminal is a , and the variables are A, B, C, D, I .

$$\begin{array}{ll}
 S \rightarrow AD & aZ \rightarrow Za \\
 A \rightarrow AI & IZ \rightarrow ZIa \\
 A \rightarrow BX & aI \rightarrow Ia \\
 XI \rightarrow IYX & BZ \rightarrow B \\
 XD \rightarrow D & B \rightarrow C \\
 YI \rightarrow IY & CI \rightarrow C \\
 YD \rightarrow ZD & C \rightarrow \lambda \\
 YZ \rightarrow ZY & D \rightarrow \lambda
 \end{array}$$

Explanation. Initially, the start symbol generates left and right bookends, A and D .

During the first phase, A generates n I 's, then changes to BX . There will never be more than one X .

During the second phase, each X pass over each I , generating a Y each time. Thus, there will be n Y 's. The X will be absorbed by the D .

During the third phase, each Y moves to the right, until it reaches D . During that movement it might need to pass over an I . Each Y changes to Z when it reaches D .

During the third phase, all Z 's move to the left, eventually being absorbed by B . Each time a Z passes over an I , a is generated. Z must be able to exchange with an a .

During the fourth phase, all I 's move to the left. To prevent any I from being absorbed before all Z 's are absorbed, the left bookend changes from B to C .

During fifth phase, both bookends change to λ .

Examples. Derivation for λ , that is, where $n = 0$:

$$S \Rightarrow AD \Rightarrow BXD \Rightarrow BD \Rightarrow CD \Rightarrow D \Rightarrow \lambda$$

Derivation for a , that is, where $n = 1$:

$$S \Rightarrow AD \Rightarrow AID \Rightarrow BXID \Rightarrow BIYXD \Rightarrow BIYD \Rightarrow BIZD \Rightarrow BZIaD \Rightarrow BIaD \Rightarrow CIaD \Rightarrow CaD \Rightarrow aD \Rightarrow a$$

Derivation for $aaaa$, that is, where $n = 2$:

$$\begin{aligned}
 S &\Rightarrow AD \Rightarrow AID \Rightarrow AIID \Rightarrow BXIID \Rightarrow BIYXID \Rightarrow BIYIYXD \Rightarrow BIYIYD \Rightarrow BIIYYD \Rightarrow \\
 &BIIYZD \Rightarrow BIIZYD \Rightarrow BIIZZD \Rightarrow BIZIaZD \Rightarrow BZIaIaZD \Rightarrow BIaIaZD \Rightarrow BIaIZaD \Rightarrow \\
 &BIaZIaaD \Rightarrow BIZaIaaD \Rightarrow BZIaaIaaD \Rightarrow BIaaIaaD \Rightarrow CIaaIaaD \Rightarrow CIaIaaaD \Rightarrow \\
 &CIIaaaaD \Rightarrow CIaaaaD \Rightarrow CaaaaD \Rightarrow aaaaD \Rightarrow aaaa
 \end{aligned}$$