**General Grammars**

The following grammar generates \( L = \{ a^{n^2} : n \geq 0 \} \). The only terminal is \( a \), and the variables are \( A, B, C, D, I \).

\[
S \rightarrow AD \\
A \rightarrow AI \\
A \rightarrow BX \\
XI \rightarrow IYX \\
XD \rightarrow D \\
YI \rightarrow IY \\
YD \rightarrow ZD \\
YZ \rightarrow ZY \\
aZ \rightarrow Za \\
IZ \rightarrow ZIa \\
aI \rightarrow Ia \\
BZ \rightarrow B \\
XD \rightarrow D \\
Y \rightarrow \lambda \\
D \rightarrow \lambda 
\]

**Explanation.** Initially, the start symbol generates left and right bookends, \( A \) and \( D \). During the first phase, \( A \) generates \( n \) \( I \)'s, then changes to \( BX \). There will never by more than one \( X \).

During the second phase, each \( X \) pass over each \( I \), generating a \( Y \) each time. Thus, there will be \( n \) \( Y \)'s. The \( X \) will be absorbed by the \( D \).

During the third phase, each \( Y \) moves to the right, until it reaches \( D \). During that movement it might need to pass over an \( I \). Each \( Y \) changes to \( Z \) when it reaches \( D \).

During the third phase, all \( Z \)'s move to the left, eventually being absorbed by \( B \). Each time a \( Z \) passes over an \( I \), an \( a \) is generated. \( Z \) must be able to exchange with an \( a \).

During the fourth phase, all \( I \)'s move to the left. To prevent any \( I \) from being absorbed before all \( Z \)'s are absorbed, the left bookend changes from \( B \) to \( C \).

During fifth phase, both bookends change to \( \lambda \).

**Examples.** Derivation for \( \lambda \), that is, where \( n = 0 \):

\[
S \Rightarrow AD \Rightarrow BXD \Rightarrow BD \Rightarrow CD \Rightarrow D \Rightarrow \lambda 
\]

Derivation for \( a \), that is, where \( n = 1 \):

\[
S \Rightarrow AD \Rightarrow AID \Rightarrow BXID \Rightarrow BIYXD \Rightarrow BIYD \Rightarrow BIZD \Rightarrow BZIaD \Rightarrow BIaD \Rightarrow CIaD \Rightarrow CaD \Rightarrow aD \Rightarrow a 
\]

Derivation for \( aaaa \), that is, where \( n = 2 \):

\[
S \Rightarrow AD \Rightarrow AID \Rightarrow AIIID \Rightarrow BXIID \Rightarrow BIYXID \Rightarrow BIYIXD \Rightarrow BIYIYD \Rightarrow BIYYD \Rightarrow BIYZD \Rightarrow BIIZYD \Rightarrow BIIZZD \Rightarrow BIZIaZD \Rightarrow BZIa1aZD \Rightarrow BIa1aZD \Rightarrow B1a1ZaD \Rightarrow B1aZIaaD \Rightarrow B1ZaIaaD \Rightarrow BZ1a1aD \Rightarrow B1aa1aaD \Rightarrow C1aa1aaD \Rightarrow C1a1aa1aaD \Rightarrow C1Ia1aa1aaD \Rightarrow CI1aa1aa1aaD \Rightarrow C1aaa1aa1aaD \Rightarrow C1aaa1aa1aaD \Rightarrow C1aaa1aa1aaD \Rightarrow C1aa1aa1aaD \Rightarrow C1aa1aa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD \Rightarrow C1aa1aaa1aaD
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