Maximum Contiguous Subsequence

Recall problem 6.1 on page 177 of your textbook. Let \( a_1, \ldots, a_n \) be the sequence, which may include both positive and negative numbers. The problem is to find the maximum sum of any contiguous subsequence.

Let \( C_k \) be the maximum of zero, and the maximum sum of any contiguous subsequence of \( a_1, \ldots, a_k \), and let \( B_k \) be the maximum of zero, and the maximum sum of any contiguous subsequence that ends at \( a_k \). We let \( B_0 = C_0 = 0 \) by default.

The problem can be reduced to the maximum path problem in a weighted layered directed graph, as shown below. In our reduction, each layer has up to three nodes, \( A[k-1], B[k], \) and \( Z \), which is always zero.

We illustrate the layered graph obtained from the input sequence 3, −1, 2, −3, −2, 3, 1, −2, 5, −3, 2 of length \( n = 11 \). The red numerals are the solution to the single source maximum path problem, and the green line indicates the maximum path to \( A_n \), which shows that the solution is the contiguous subsequence 3, 1, −2, 5, whose total is 7.

Parallel Computation

Instead of using dynamic programming, we can solve the maximum path problem by \((\text{max, +})\) matrix multiplication. Each layer is represented by a vector of length 1, 2, or 3. The first of those vectors is \((0)\). Each subsequent vector is the \((\text{max, +})\) product of the previous vector by a matrix. The product of these matrices can be computed in parallel by \( n \) processors in \( O(\log n) \) time, and then multiplied by the initial vector.

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 & 3 & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\infty & -\infty \\
-\infty & 0 & 0
\end{pmatrix}
= (7)
\]

In the example, the product of the 13 matrices is \((7)\); applying to the initial vector: \((0)(7) = (7)\).