University of Nevada, Las Vegas Las Vegas Computer Science 477/677 Fall 2019

Answers to Assignment 3: Due Wednesday September 11, 2019

1. Work Problem 2.4 on page 71 of your textbook. In each case, let n be the size of the original problem.

   - The time for algorithm A satisfies the reucurrence $T(n) = 5T(n/2) + \Theta(n)$. The solution to this recurrence is $T(n) = \Theta\big(n^{\log_2 5}\big)$.
   - The time for algorithm B satisfies the reucurrence $T(n) = 2T(n-1) + \Theta(1)$. The solution to this recurrence is $T(n) = \Theta(2^n)$.
   - The time for algorithm C satisfies the reucurrence $T(n) = 9T(n/3) + \Theta(n^2)$. The solution to this recurrence is $T(n) = \Theta\big(n^2\big)\log n$, since $\log_3 9 = 2$.

   Algorithm C is fastest, since $\log_2 5 > 2$.

2. The following problem is similar to problem 3.5 on page 71 of in your textbook. Solve the followring recurrences. Give a Θ bound if possible; otherwise if an Ω or an O bound.

   (a) $T(n) \leq 5T(n/5) + 1$.
   $T(n) = O(n)$.

   (b) $T(n) = 5T(n/5) + n$.
   $T(n) = \Theta(n \log n)$.

   (c) $T(n) \geq 5T(n/5) + n^2$. $T(n) = \Omega(n^2)$.

   (d) $T(n) = 25T(n/5) + n^2$. $T(n) = \Theta(n^2 \log n)$.

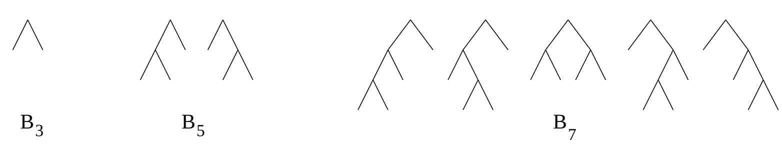
   (e) $T(n) \leq T(n-1) + n^4$.
   $T(n) = O(n^5)$.

   (f) $T(n) = 3T(n-1) + 1$.
   $T(n) = \Theta(3^n)$.

   (g) $T(n) \geq T(\sqrt{n}) + 1$. Let $T(n) = F(\ell)$ where $\ell = \log n$. (The base of the logarithm doesn't matter.) $T(\sqrt{n}) = F(\log(\sqrt{n})) = F(\log n/2) = F(\ell/2)$. Rewriting the recurrence, we have $F(\ell) \geq F(\ell/2)+1$, hence $T(n) = F(\ell) = \Omega(\log \ell) = \Omega(\log\log n)$.

3. Work problem 2.13(a,b) in your textbook.

   (a) $B_3 = 1$, $B_5 = 2$, and $B_7 = 5$.

   $B_n = 0$ if n is even, because a full binary tree must have an odd number of vertices.

(b) We have

$B_1 = 1.$

$B_3 = B_1B_1 = 1$

$B_5 = B_3B_1 + B_1B_3 = 2$

$B_7 = B_5B_1 + B_3B_3 + B_1B_5 = 5$

In general $B_n = \sum_{i=1}^{n/2} B_{n-2i}B_{2i-1}$ for n odd, $n \geq 3$.

The *Catalan* numbers are $1, 1, 2, 5, 14, 42, \ldots$ and B_{2n+1} is the n^{th} Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

4. Work problem 2.16 in you textbook.

Let the array have indices starting with zero, as in $C++$ Our first phase sets `lo = 0` and `hi` to be some integer such that `A[hi] > x`. We initialize `hi = 1`, and keep doubling `i` until `A[hi] > x`. Our loop invariant is that either `A[i] = x` for some `lo <= i < hi` or `x` is not an entry of the array.

The second phase is to use binary search to halve the size of the search inteveral at each step, until `hi = lo+1`. At this point, since the loop invariant still holds, either `A[lo] = x` or `x` is not in the array.

```
int lo = 0;
int hi = 1;
while(A[hi] <= x) hi = 2*hi;
while (lo+1 < hi)
 {
  int mid = (lo+hi)/2;
  if(A[mid] <= x) lo = mid; // loop invariant is maintained
  else hi = mid; // loop invariant is maintained
 }
if(A[lo] == x)
 cout << "A[" << lo << "] = " << x << endl;
else
 cout << x << " is not an entry in the array" << endl;
```

Each phase takes $O(\log n)$ steps, hence the time complexity of the algorithm is $O(\log n)$.

5. Work problem 2.22 in you textbook.

Mr. Singh, I have deleted my work on this problem. The basic idea is not very difficult to understand, but the details are a killer. Hopefully, I can finish those details soon.

6. If $f(n)$ is an increasing function, We say that f is *polylogarithmic* if $\log(f(n)) = \Theta(\log\log n)$. We say that f is *polynomial* if $\log(f(n)) = \Theta(\log n)$. We say that f is *exponential* if $\log(f(n)) = \Theta(n)$.

It turns out that not every increasing function falls into one of those classes. Suppose F satisfies the recurrence:

$$F(n) = F(n/2) + F(n-1) + 1$$

It is obvious that $n < F(n) < 2^n$, so F grows at least as fast as polynomial but no faster than exponential.

(a) Is F polylogarithmic? (Hint: No.)

(b) Is F polynomial? No. F grows faster than any polynomial function.

(c) Is F exponential? No. F grows slower than any exponential function.

Not every polynomial function (as defined above) is $\Theta(n^K)$ for some constant $K > 1$, and not every exponential function (as defined above) is $\Theta(2^{Cn})$ for some constant $C > 0$. However, these simplifications are "almost" true: more specifically, every polynomial function is both $O(n^{K_1})$ and $\Omega(n^{K_2})$ for constants $K_1 \geq K_2 > 1$, while every exponential function is both $O(2^{C_1 n})$ and $\Omega(2^{C_2 n})$ for constants $C_1 \geq C_2 > 0$.

We first note that $F(n) \geq F(n-1) + 1$, hence F is monotone increasing. The formula that defines F works when you substitute any quantity for n, such as $n-1$, $n-2$, etc. Thus

$$\begin{aligned}
F(n) &= F(n/2) + F(n-1) + 1 \\
F(n-1) &= F((n-1)/2) + F(n-2) + 1 \\
F(n-2) &= F((n-2)/2) + F(n-3) + 1 \\
& \quad \text{and so forth. Substituting, we obtain} \\
F(n) &= F(n/2) + F((n-1)/2) + F((n-2)/2) + F(n-3) + 3 \\
& \quad \text{Repeated substitution yields, for any } m \\
F(n) &= F(n/2) + F((n-1)/2) + \cdots + F((n-m+1)/2) + F(n-m) + m \\
& \quad \text{Let } m = n/2 \text{ (assuming that is an integer)} \\
F(n) &= F(n/2) + F((n-1)/2) + \cdots + F((n/2+1)/2) + F(n/2) + n/2 \\
& \quad \text{Since } F \text{ is monotone increasing, we have two inequalities} \\
F(n) &\geq nF(n/4)/2 + F(n/2) + n/2 \\
F(n) &\leq nF(n/2)/2 + F(n/2) + n/2
\end{aligned}$$

To make the problem easier to understand, We make the simplifying assumption that a polynomial function of n is of the form n^K for some constant K, and that an exponential function of n is of the form 2^{Cn} for some constant C.

Suppose $F(n) = n^K$ for some constant K. Then

$$\begin{aligned}
F(n) &\geq nF(n/4) \\
n^K &\geq \frac{n^{K+1}}{4^K} \\
4^K &\geq n
\end{aligned}$$

Constradiction, since 4^K is constant and n is arbitrarily large.

Suppose $F(n) = 2^{Cn}$ for some constant C. Then

$$\begin{aligned}
2^{Cn} &\leq 2^{Cn/2}n + 2^{Cn/2} + n/2 \\
& \quad \text{divide both sides by } 2^{Cn/2} \\
2^{Cn/2} &\leq (n+1) + \frac{n}{2^{Cn/2-1}}
\end{aligned}$$

Contradiction, since an exponential function grows faster than a polynomial function.