The Complex Number System

In $\mathbb{R}$, the real number system, a negative number has no square root. We can fill in that gap by inventing a new number system $\mathbb{C}$, the complex numbers, also called the Gaussian numbers. (After Karl Friedrich Gauss.) In $\mathbb{C}$, every number except zero has two square roots, three cube roots, four 4^{th} roots, *etc.*.

A complex number is represented in two standard ways: Cartesian or Polar. If z is a complex number, we can write $z = a + b\,\mathbf{i}$, where $a = Re(z)$, the *real part* of z, and $b = Im(z)$, the *imaginary part* of z. Note that the imaginary part of a complex number is a real number.

Arithmetic with Complex Numbers

All the arithmetic operations on complex numbers expressed in $a + b\,\mathbf{i}$ (Cartesion) form are computed using high school algebra, plus just one additional rule: $\mathbf{i}^2 = -1$.

Let $z = a + b\,\mathbf{i}$ and $w = c + d\,\mathbf{i}$, where a, b, c, d are real numbers.

1. Addition: $z + w = (a + c) + (b + d)\,\mathbf{i}$.
2. Subtraction: $z - w = (a - c) + (b - d)\,\mathbf{i}$.
3. Multiplication: $z * w$, or simply zw, is $(ab - bd) + (ad + bc)\,\mathbf{i}$.
4. Conjugate: $\overline{z} = a - b\,\mathbf{i}$.
5. Modulus: $|z| = \sqrt{a^2 + b^2}$, also called the *absolute value* of z. Note that $z\overline{z} = |z|^2$.
6. Division: $z/w = \dfrac{z}{w} = \dfrac{z\overline{w}}{w\overline{w}} = \dfrac{(a + b\,\mathbf{i})(c - d\,\mathbf{i})}{(c + d\,\mathbf{i})(c - d\,\mathbf{i})} = \dfrac{(ac + bd) + (-ad + bc)\,\mathbf{i}}{c^2 + d^2}$
7. Square Root: $$\sqrt{z} = \pm \left\{ \begin{array}{l} \sqrt{\dfrac{\sqrt{a^2 + b^2} + a}{2}} + \sqrt{\dfrac{\sqrt{a^2 + b^2} - a}{2}}\,\mathbf{i} \text{ if } b > 0 \\ \sqrt{\dfrac{\sqrt{a^2 + b^2} + a}{2}} - \sqrt{\dfrac{\sqrt{a^2 + b^2} - a}{2}}\,\mathbf{i} \text{ if } b < 0 \end{array} \right\}$$

   Note that the square root sign, $\sqrt{\ }$ indicates the positive square root of a positive real number.

Cube roots of $a + b\,\mathbf{i}$ do not have a convenient representation.

Arithmetic with Complex Numbers in Polar Form

Let $z = \varrho\,(\cos\theta + \mathbf{i}\sin\theta)$ and $w = \varsigma\,(\cos\phi + \mathbf{i}\sin\phi)$

1. Addition and Subtraction do not have convenient polar representations.
2. Argument: $\arg z = \theta$. But we can also say $\arg z = \theta + 2k\pi$ for any integer k. Arguments are angles, measured in either degrees or radians. 360 degrees equals 2π radians, since the circumference of a circle is 2π times the radius.
3. Conjugate: $\overline{z} = \varrho(\cos(-\theta) + \mathbf{i}\sin(-\theta)$. That is, $\arg\overline{z} = -\arg z$.
4. Modulus: $|z| = \varrho$.
5. Multiplication: Take the sum of the arguments and the product of the moduli.
   $zw = \varrho\varsigma(\cos(\theta+\phi) + \mathbf{i}\varsigma\sin(\theta+\phi))$.
6. Division: Take the difference of the arguments and the quotient of the moduli.
   $z/w = \frac{\varrho}{\varsigma}(\cos(\theta-\phi) + \mathbf{i}\sin(\theta-\phi))$
7. Square Root: Take half the argument and the square root of the modulus.
   $\sqrt{z} = \pm(\sqrt{\varrho}(\cos\frac{\theta}{2} + \mathbf{i}\sin\frac{\theta}{2})$
8. Cube Root: Take a third of the argument and the cube root of the modulus. But there are three cube roots:

$$\sqrt[3]{z} = \begin{cases} (\sqrt[3]{\varrho}(\cos\frac{\theta}{3} + \mathbf{i}\sin\frac{\theta}{3}) \\ (\sqrt[3]{\varrho}(\cos\frac{\theta+2\pi}{3} + \mathbf{i}\sin\frac{\theta+2\pi}{3}) \\ (\sqrt[3]{\varrho}(\cos\frac{\theta+4\pi}{3} + \mathbf{i}\sin\frac{\theta+4\pi}{3}) \end{cases}$$

Conversion to Polar Representation

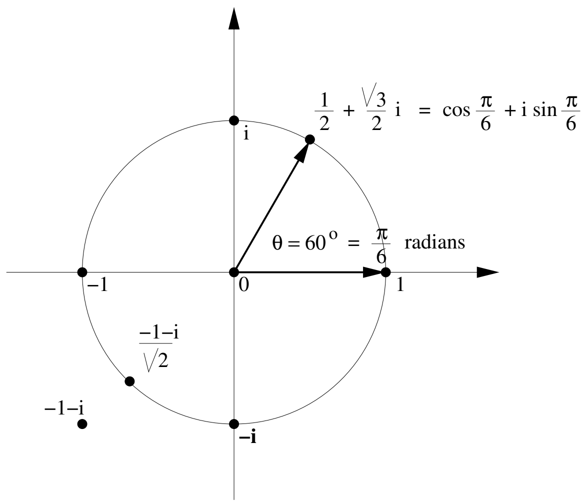
If $z = a + b\,\mathbf{i}$, we would like to find ϱ and θ such that $z = \varrho(\cos\theta + \mathbf{i}\sin\theta)$. $\varrho = |z| = \sqrt{a^2+b^2}$. If $a \geq 0$, we can choose $\theta = \arctan\frac{b}{a}$. How would you find θ if $a \leq 0$?

The Exponential Function on $\mathbb{C}$

Recall $e \approx 2.71828\ldots$, the base of the natural logarithm. e^z is defined for any complex number z. If $z = a+bi$, the modulus of e^z is e^a, while the argument of e^z is b. That is: $e^z = e^a(\cos b + \mathbf{i}\sin b)$.

Roots of Unity

The word "unity" means the complex number $1+0\,\mathbf{i}$, or simply 1. For any n, the n^{th} roots of unity are the n solutions to the equation $z^n = 1$, which are written either as $e^{\frac{2k\pi\,\mathbf{i}}{n}}$, or $\cos\frac{2k\pi}{n} + \mathbf{i}\sin\frac{2k\pi}{n}$ for $k = 0, 1, \ldots n-1$. For example, the 4 4^{th} roots of unity are 1, $\mathbf{i}$, -1, $-\,\mathbf{i}$. The 3 cube roots of unity are 1, $\frac{-1+\sqrt{3}\,\mathbf{i}}{2}$, $\frac{-1-\sqrt{3}\,\mathbf{i}}{2}$. Roots of unity always lie on the unit circle centered at zero.



What is the modulus of $-1-\mathbf{i}$? Ans: $\sqrt{2}$. What is the argument of $-1-\mathbf{i}$? Ans: 225 degrees, or $\frac{5\pi}{4}$ radians. Radian measure is the default, so we can simply say the argument is $\frac{5\pi}{4}$. It would also be correct to say the argument is -135 degrees or $-\frac{3\pi}{2}$.