A Short History of Complex Numbers

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Abstract

This is a compilation of historical information from various sources, about the number $i = \sqrt{-1}$. The information has been put together for students of Complex Analysis who are curious about the origins of the subject, since most books on Complex Variables have no historical information (one exception is Visual Complex Analysis, by T. Needham).

A fact that is surprising to many (at least to me!) is that complex numbers arose from the need to solve *cubic* equations, and not (as it is commonly believed) quadratic equations. These notes track the development of complex numbers in history, and give evidence that supports the above statement.

1. Al-Khwarizmi (780-850) in his Algebra has solution to quadratic equations of various types. Solutions agree with is learned today at school, restricted to positive solutions [9] Proofs are geometric based. Sources seem to be greek and hindu mathematics. According to G. J. Toomer, quoted by Van der Waerden,

   > Under the caliph al-Ma'mun (reigned 813-833) al-Khwarizmi became a member of the "House of Wisdom" (Dar al-Hikma), a kind of academy of scientists set up at Baghdad, probably by Caliph Harun al-Rashid, but owing its preeminence to the interest of al-Ma'mun, a great patron of learning and scientific investigation. It was for al-Ma'mun that Al-Khwarizmi composed his astronomical treatise, and his Algebra also is dedicated to that ruler

2. The methods of algebra known to the arabs were introduced in Italy by the Latin translation of the algebra of al-Khwarizmi by Gerard of Cremona (1114-1187), and by the work of Leonardo da Pisa (Fibonacci)(1170-1250).

   About 1225, when Frederick II held court in Sicily, Leonardo da Pisa was presented to the emperor. A local mathematician posed several problems, all of which were solved by Leonardo. One of the problems was the solution of the equation

   $$x^3 + 2x^2 + 10x = 20$$

3. The general cubic equation

   $$x^3 + ax^2 + bx + c = 0$$

   can be reduced to the simpler form

   $$x^3 + px + q = 0$$

through the change of variable $x' = x + \frac{1}{3}a$. This change of variable appears for the first time in two anonymous florentine manuscripts near the end of the 14th century.

If only positive coefficients and positive values of x are admitted, there are three cases, all collectively known as *depressed cubic*:

$$\begin{array}{lrcl} (a) & x^3 + px & = & q \\ (b) & x^3 & = & px + q \\ (c) & x^3 + q & = & px \end{array}$$

4. The first to solve equation (1) (and maybe (2) and (3)) was Scipione del Ferro, professor of U. of Bologna until 1526, when he died. In his deathbed, del Ferro confided the formula to his pupil Antonio Maria Fiore. Fiore challenged Tartaglia to a mathematical contest. The night before the contest, Tartaglia rediscovered the formula and won the contest. Tartaglia in turn told the formula (but not the proof) to Gerolamo Cardano, who signed an oath to secrecy. From knowledge of the formula, Cardano was able to reconstruct the proof. Later, Cardano learned that del Ferro had the formula and verified this by interviewing relatives who gave him access to del Ferro's papers. Cardano then proceeded to publish the formula for all three cases in his Ars Magna (1545). It is noteworthy that Cardano mentioned del Ferro as first author, and Tartaglia as obtaining the formula later in independent manner.

5. A difficulty in case (2) that was not present in the solution to (1) is the possibility of having the square root of a negative number appear in the numerical expression given by the formula. Here is the derivation: Substitute $x = u + v$ into $x^3 = px + q$ to obtain

   $$x^3 - px = u^3 + v^3 + 3uv(u + v) - p(u + v) = q$$

   Set $3uv = p$ above to obtain $u^3 + v^3 = q$ and also $u^3v^3 = (p/3)^3$. That is, the sum and the product of two cubes is known. This is used to form a quadratic equation which is readily solved:

   $$x = u + v = \sqrt[3]{\frac{1}{2}q + w} + \sqrt[3]{\frac{1}{2}q - w}$$

   where

   $$w = \sqrt{(\frac{1}{2}q)^2 - (\frac{1}{3}p)^3}$$

   The so-called *casus irreducibilis* is when the expression under the radical symbol in w is negative. Cardano avoids discussing this case in Ars Magna. Perhaps, in his mind, avoiding it was justified by the (incorrect) correspondence between the *casus irreducibilis* and the lack of a real, positive solution for the cubic.

6. According to [9], "Cardano was the first to introduce complex numbers $a + \sqrt{-b}$ into algebra, but had misgivings about it." In Chapter 37 of Ars Magna the following problem is posed: "To divide 10 in two parts, the product of which is 40".

   > It is clear that this case is impossible. Nevertheless, we shall work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book leaving a remainder of -15, the square

root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5+\sqrt{-15}$ and $5-\sqrt{-15}$.

Putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ and $5-\sqrt{-15}$ making $25-(-15)$ which is $+15$. Hence this product is 40.

7. Rafael Bombelli authored l'Algebra (1572, and 1579), a set of three books. Bombelli introduces a notation for $\sqrt{-1}$, and calls it "*piú di meno*".

   The discussion of cubics in l'Algebra follows Cardano, but now the casus irreducibilis is fully discussed. Bombelli considered the equation

   $$x^3 = 15x + 4$$

   for which the Cardan formula gives

   $$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

   Bombelli observes that the cubic has $x = 4$ as a solution, and then proceeds to explain the expression given by the Cardan formula as another expression for $x = 4$ as follows. He sets

   $$\sqrt[3]{2+\sqrt{-121}} = a + bi$$

   from which he deduces

   $$\sqrt[3]{2-\sqrt{-121}} = a - bi$$

   and obtains, after algebraic manipulations, $a = 2$ and $b = 1$. Thus

   $$x = a + bi + a - bi = 2a = 4$$

   After doing this, Bombelli commented:

   > " At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof."

8. René Descartes (1596-1650) was a philosopher whose work, La Géométrie, includes his application of algebra to geometry from which we now have Cartesian geometry. Descartes was pressed by his friends to publish his ideas, and he wrote a treatise on science under the title "Discours de la méthod pour bien conduire sa raison et chercher la vérité dans les sciences". Three appendices to this work were La Dioptrique, Les Météores, and La Géométrie. The treatise was published at Leiden in 1637. Descartes associated imaginary numbers with geometric impossibility. This can be seen from the geometric construction he used to solve the equation $z^2 = az - b^2$, with a and b^2 both positive. According to [1], Descartes coined the term *imaginary*:

   > "For any equation one can imagine as many roots [as its degree would suggest], but in many cases no quantity exists which corresponds to what one imagines."

9. John Wallis (1616-1703) notes in his *Algebra* that negative numbers, so long viewed with suspicion by mathematicians, had a perfectly good physical explanation, based on a line with a zero mark, and positive numbers being numbers at a distance from the zero point to the right, where negative numbers are a distance to the left of zero. Also, he made some progress at giving a geometric interpretation to $\sqrt{-1}$.

10. Abraham de Moivre (1667-1754) left France to seek religious refuge in London at eighteen years of age. There he befriended Newton. In 1698 he mentions that Newton knew, as early as 1676 of an equivalent expression to what is today known as de Moivre's theorem:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

where n is an integer. Apparently Newton used this formula to compute the cubic roots that appear in Cardan formulas, in the irreducible case. de Moivre knew and used the formula that bears his name, as it is clear from his writings -although he did not write it out explicitly.

11. L. Euler (1707-1783) introduced the notation $i = \sqrt{-1}$ [3], and visualized complex numbers as points with rectangular coordinates, but did not give a satisfactory foundation for complex numbers. Euler used the formula $x + iy = r(\cos\theta + i\,\sin\theta)$, and visualized the roots of $z^n = 1$ as vertices of a regular polygon. He defined the complex exponential, and proved the identity $e^{i\theta} = \cos\theta + i\,\sin\theta$.

12. Caspar Wessel (1745-1818), a Norwegian, was the first one to obtain and publish a suitable presentation of complex numbers. On march 10, 1797, Wessel presented his paper "On the Analytic Representation of Direction: An Attempt" to the Royal Danish Academy of Sciences. The paper was published in the Academy's Memoires of 1799. Its quality was judged to be so high that it was the first paper to be accepted for publication by a non-member of the academy.

    Wessel's paper, written in danish, went unnoticed until 1897, when it was unearthed by an antiquarian, and its significance recognized by the Danish mathematician Sophus Christian Juel.

    Wessel's approach used what we today call vectors. He uses the geometric addition of vectors (parallelogram law) and defined multiplication of vectors in terms of what we call today adding the polar angles and multiplying the magnitudes.

13. Jean-Robert Argand (1768-1822) was a Parisian bookkeeper. It is not known whether he had mathematical training. Argand produced a pamphlet in 1806, run by a private press in small print. He failed to include his name in the title page. The title was "Essay on the Geometrical Interpretation of Imaginary Quantities". One copy ended up in the hands of the mathematician A. Legendre (1752-1833) who in turn mentioned it in a letter to Francois Francais, a professor of mathematics. When Francais died, he inherited his papers to his brother Jaques who was a professor of military art and a mathematician. He found Legendre's letter describing Argand's mathematical results, but Legendre failed to mention Argand. Jaques published an article in 1813 in the Annales de Mathémathiques giving the basics of complex numbers. In the last paragraph of the paper, Jaques acknowledged his debt to Legendre's letter, and urged the unknown author to come forward. Argand learned of this and his reply appeared in the next issue of the journal.

14. William Rowan Hamilton (1805-65) in an 1831 memoir defined *ordered pairs* of real numbers (a, b) to be a *couple*. He defined addition and multiplication of couples: $(a, b)+(c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, bc + ad)$. This is in fact an algebraic definition of complex numbers.

15. Carl Friedrich Gauss (1777-1855). There are indications that Gauss had been in possession of the geometric representation of complex numbers since 1796, but it went unpublished until 1831, when he submitted his ideas to the Royal Society of Gottingen. Gauss introduced the term *complex number*

    > "If this subjet has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had +1, -1 and $\sqrt{-1}$, instead of being called positive, negative and imaginary (or worse still, impossible) unity, been given the names say,of direct, inverse and lateral unity, there would hardly have been any scope for such obscurity."

    In a 1811 letter to Bessel, Gauss mentions the theorem that was to be known later as Cauchy's theorem. This went unpublished, and was later rediscovered by Cauchy and by Weierstrass.

16. Augustin-Louis Cauchy (1789-1857) initiated complex function theory in an 1814 memoir submitted to the French Académie des Sciences. The term analytic function was not mentioned in his memoir, but the concept is there. The memoir was published in 1825. Contour integrals appear in the memoir, but this is not a first, apparently Poisson had a 1820 paper with a path not on the real line. Cauchy constructed the set of complex numbers in 1847 as $R[x]/(x^2+1)$

    > "We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it."

References

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[4] P. Nahin, *An Imaginary Tale*, Princeton U. Press, NJ 1998.

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[6] The MacTutor History of Mathematics archive, at `www.gap-system.org/ history/`

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[9] B. L. van der Waerden, *A History of Algebra*, Springer Verlag, NY 1985.