# Recurrences: Akra Brazzi Method

We consider recurrences of the following form:

$F(n) = \sum_{i=1}^{k} a_i F(b_i n) + n^c$

where

- $a_i > 0$ for each $i$
- $0 < b_i < 1$ for each $i$
- $c \geq 0$

First, compute $\sum_{i=1}^{k} a_i b_i^c$. There are three formulae for the solution.

Case 1: $\sum_{i=1}^{k} a_i b_i^c < 1$

$F(n) = \Theta(n^c)$

Case 2: $\sum_{i=1}^{k} a_i b_i^c = 1$

$F(n) = \Theta(n^c \log n)$

Case 3: $\sum_{i=1}^{k} a_i b_i^c > 1$

First, find an exponent $d$ such that $\sum_{i=1}^{k} a_i b_i^d = 1$. Then

$F(n) = \Theta(n^d)$.

## Examples

(i) $F(n) = F(n/2) + F(n/3) + F(n/6) + n^2$
$a_1 = 1, b_1 = \frac{1}{2}, a_2 = 1, b_2 = \frac{1}{3}, a_3 = 1, b_3 = \frac{1}{6}, c = 2$
$a_1 b_1^c + a_2 b_2^c + a_3 b_3^c = \frac{1}{4} + \frac{1}{9} + \frac{1}{36} = \frac{7}{18} < 1$
Case 1: $F(n) = \Theta(n^c) = \Theta(n^2)$

(ii) $F(n) = F(n/2) + F(n/3) + F(n/6) + n$
$a_1 = 1, b_1 = \frac{1}{2}, a_2 = 1, b_2 = \frac{1}{3}, a_3 = 1, b_3 = \frac{1}{6}, c = 1$
$a_1 b_1^c + a_2 b_2^c + a_3 b_3^c = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$
Case 2: $F(n) = \Theta(n^c \log n) = \Theta(\log n)$

(iii) $F(n) = F(n/2) + F(n/3) + F(n/6) + 1$

$a_1 = 1, b_1 = \frac{1}{2}, a_2 = 1, b_2 = \frac{1}{3}, a_3 = 1, b_3 = \frac{1}{6}, c = 0$

$a_1 b_1^c + a_2 b_2^c + a_3 b_3^c = 1 + 1 + 1 = 3 > 1$

Case 3: Find $d$ such that $a_1 b_1^d + a_2 b_2^d + a_3 b_3^d = 1$.

$d = 1$

$F(n) = \Theta(n^d) = \Theta(n)$

(iv) $F(n) = F(n/9) + F(4n/9) + \sqrt{n}$

$a_1 = 1, b_1 = \frac{1}{9}, a_2 = 1, b_2 = \frac{4}{9}, c = \frac{1}{2}$

$a_1 b_1^c + a_2 b_2^c = \frac{1}{3} + \frac{2}{3} = 1$

Case 2: $F(n) = \Theta(n^c \log n) = \Theta(\sqrt{n} \log n)$

(v) $F(n) = 3F(n/3) + 3F(2n/3) + n^2$

$a_1 = 3, b_1 = \frac{1}{3}, a_2 = 3, b_2 = \frac{2}{3}, c = 2$

$a_1 b_1^c + a_2 b_2^c = 3(\frac{1}{3})^2 + (\frac{2}{3})^2 = \frac{5}{9} < 1$

Case 3: Find $d$ such that $a_1 b_1^d + a2 b_2^d = 1$. $d = 3$

$F(n) = \Theta(n^d) = \Theta(n^3)$