University of Nevada, Las Vegas Computer Science 477/677 Spring 2020

Answers to Examination April 30, 2020

Name:____

The exam is take-home, open book, open notes, open internet. You must finish by midnight of April 30. Scan and email the completed examination paper to your TA, Shekhar Singh. The email must have an April 30 time stamp.

The entire examination is 285 points.

1. Solve the recurrences. Give asymptotic answers in terms of n, using either O, Ω, or Θ, whichever is most appropriate. Use whichever technique is appropriate for each problem. [10 points each]

   (a) $F(n) = 2F(n/2) + n$ $\quad F(n) = \Theta(n \log n)$ by the master theorem.

   (b) $F(n) \le F(n-3) + 3\log n$ $\quad F(n) - F(n-3) = 3\log n$

   $\frac{F(n) - F(n-3)}{3} = \log n \qquad F'(n) = \Theta(\log n) \qquad F(n) = \Theta(n \log n)$

   (c) $F(n) = F(\sqrt{n}) + 1$; (Hint: use a substitution. Introduce the variable m and let $m = \log_2 n$, and introduce the function G such that $G(m) = F(2^m) = F(n)$. Then find a new recurrence using the function G. Solve that recurrence, and then substitute back.. to solve the original recurrence.)

   $G(m) = G(m/2) + 1 \qquad F(n) = G(m) = \Theta(\log m)$ by the master theorem $= \Theta(\log\log n)$

   (d) $F(n) \le F(n/2) + F(n/3) + n$ $\quad F(n) = O(n)$ by the generalized master theorem since $\frac{1}{2} + \frac{1}{3} < 1$

   (e) $F(n) \le 2F(n-1) + 1$
   Let $m = 2^n$ and $G(m) = F(n)$. Then $m/2 = 2^{n-1}$ hence $F(n-1) = G(m/2)$
   $G(m) = 2G(m/2) + 1$ $G(m) = \Theta(m)$ by the master theorem
   $F(n) = G(m) = \Theta(m) = \Theta(2^n)$

   (f) $F(n) \ge 4F(n/2) + n$ $\quad F(n) = \Omega(n^2)$ by the master theorem since $\log_2 4 = 2$

   (g) $F(n) = F(n - \sqrt{n}) + \sqrt{n}$ $\quad F(n) - F(n - \sqrt{n}) = \sqrt{n}$

   $\frac{F(n) - F(n-\sqrt{n})}{\sqrt{n}} = \Theta(1) \qquad F'(n) = \Theta(1) \qquad F(n) = \Theta(n)$

2. [10 points] Find an integer K such that the solution to the recurrence below is $F(n) = \Theta(n^2 \log n)$.

   $F(n) = F(3n/5) + K\,F(n/5) + n^2$

   By the master theorem, $\left(\frac{3}{5}\right)^2 + K\left(\frac{1}{5}\right)^2 = 1$ hence $K = 16$.

3. Give the asymptotic time complexity of each of these code fragments, in terms of n. [10 points each.]

(a) `for(int i = n; i > 1; i = log2(i)))`

Recall the recursive definition: $\log^* x = \begin{cases} 0 \text{ if } x \leq 1 \\ 1 + \log^* \log_2 x \text{ otherwise} \end{cases}$

Thus $\log^* \log_2 x = \log^* x - 1$ if $x > 1$.

Let $j = \log^* i$. We obtain

for(int j = $\log^*$n; j > 1; j = j−1)

The answer is then $\Theta(\log^* n)$.

(b)
```
for(int i = n; i > 1; i = i/2)
 for(int j = 1; j < i; j++)
```

Let $k = \log_2 i$: then $i = 2^k$. We obtain

```
for(int k = log2(n); k > 0; k = k-1)
 for(int j = 1; j < 2^k; j++)
```

We approximating by integrals, letting y, z be the continuous analogs of j, k, respectively. Conventional notation of integrals requires that the values of z must be increasing despite the fact that the values of k are decreasing in the code.

$$\int_{z=1}^{\log_2 n} \int_{y=1}^{2^z} dydz = \int_{z=1}^{\log_2 n} (2^z - 1)dz = \left(\frac{2^z}{\ln 2} - z\right)\Bigg]_1^{\log_2 n} = \Theta(n)$$

Alternative computation using summations. Conventional notation requires the value of the index k to increase, despite the fact that it decreases in the code. Recall that $1+2+4+\cdots+2^m = 2^{m+1}-1$.

$$\sum_{k=0}^{\log_2 n-1} 2^k = 2^{\log_2 n} - 1 = n - 1 = \Theta(n)$$

(c)
```
for(int i = n; i > 1; i = i/2)
 for(int j = i; j < n; j++)
```

Using the same substitution $k = \log_2 i$ and using the summation method:

$$\sum_{k=0}^{\log_2 n-1} (n - 2^k) = n(\log_2 n) - 2^{\log_2 n} = kn - (n-1) = \Theta(n \log n)$$

(d)
```
for(int i = 0; i < n; i++)
 for(int j = 0; j < i*i; j++)
```

Recall that $1 + 4 + 9 + \cdots m^2 = \dfrac{m(m+1)(2m+1)}{6}$

$$\sum_{i=0}^{n-1} i^2 = \Theta\big(n^3\big)$$

We can use integration instead:

$$\sum_{i=0}^{n-1} i^2 = \Theta\left(\int_{x=0}^{n-1} x^2 dx\right) = \Theta\left(\frac{(n-1)^3}{3}\right) = \Theta(n^3)$$

(e) `for(int i = 2; i < n; i = i*i)`

Let $j = \log_2 i$

```
for(int j = 1; j < log2(n); j = 2*j)
```

Let $k = \log_2 j$

```
for(int k = 0; k < log2(log2(n)); k = k+1)
```

$\Theta(\log\log n)$

(f) `for(int i = n; i > 1; i = sqrt(i))`

Let $j = \log_2 i$

```
for(int j = log2(n); j > 0; j = j/2)
```

Let $k = \log_2 j$

```
for(int k = log2(log2(n)); k >= 0; k = k-1)
```

$\Theta(\log\log n)$

(g)
```
for(int i = n; i > 1; i = sqrt(i))
 for(int ell = 0; ell < i; ell++)
```

Since I want to use j for $\log_2 i$, I will change the second variable to ℓ. Let $j = \log_2 i$

```
for(int j = log2(n); j > 0; j = j/2)
 for(int ell = 0; ell < 2^j; ell++)
```

Let $k = \log_2 j$

```
for(int k = log2(log2(n)); k >= 0; k = k-1)
 for(int ell = 0; ell < 2^(2^k); ell++)
```

Use the summatrion method. We have $\displaystyle\sum_{k=0}^{\log_2 \log_2 n} 2^{2^k}$

The last term of the summation is greater than the sum of all the previous terms, thus the summation is $\Theta\left(2^{2^{\log_2 \log_2 n}}\right) = \Theta(n)$.

4. Arrays. There is a built-in implementation of the abstract data type Array in C++. However, sometimes it is best to use a different implemntation. In Problems 4 and 5 I was very generous in grading, since I realized the answers were not clear-cut. I won't make that mistake on the final.

   Suppose that an $n \times m$ array A has k non-zero entries. I wanted you to choose one of the following implentations:

   - Standard implemnations: one memory location for each entry.
   - Array of search structures, one for each row, an array R of length n, where $R[i]$ is a search structure which holds all ordered pairs of the form (j, x) where $A[i][j] = x$ and $x \neq 0$.
   - Search structure of non-zero entries: A search structure which holds all ordered triples (i, j, x) where $A[i][j] = x$ and $x \neq 0$.

   (a) [10 points] What space saving data structure would you use to implement a 1000×1000 2-dimensional array where there are 2000 non-zero entries, the rest zero?

   Use an array of search structures, one for the non-zero entries of each row.

   (b) [10 points] What space saving data structure would you use to implement a 10000×10000 2-dimensional array which has 100 non-zero entries, the rest zero?

   Search structure of non-zero entries.

5. Graphs.

   (a) [10 points] What does it mean to say that a graph is "sparse"?

   The term is not well-defined, but generally it means that the the number of edges in the graph is much lower than the maximum possible. If n, m are the numbers of vertices and edges, we can usually say the graph is sparse if $m = o(n^2)$.

   (b) [10 points] What data structure would you use to implement a graph with 1000 vertices and 10000 edges?

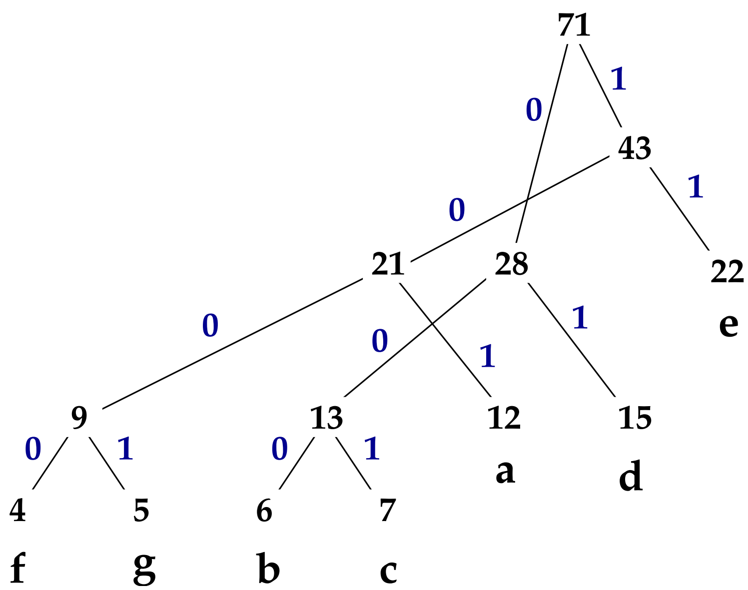
   The answer I want is, an array of search structures, one for each vertex. The entry for a vertex v is a search structure containing all neighbors of v.

   (c) [5 points] What is the maximum number of edges a planar graph with 5 vertices can have?

   The formula for a planar graph is $m \leq 3n - 6$ if $n > 2$. Thus $m \leq 9$.

6. [20 points] Construct an optimal prefix-free code for the alphabet a,b,c,d,e,f with the frequencies given by the table below.

a	12	101
b	6	000
c	7	001
d	15	01
e	22	11
f	4	1000
g	5	1001



This answer is not unique. Any prefix-free code where the code strings for d and e have length 2, for a, b, and c have length 3, and for f and g have length 4, is optimal.

7. [20 points] In my video at https://www.youtube.com/watch?v=iKA4URlAtKo I show how to implement a min-heap as an array. For example, a min-heap of size 6 could be implemented as the following array:

D	F	H	M	L	R

If *deletemin* is executed, the array changes in a sequence of steps. Show those steps.

R	F	H	M	L	R
F	R	H	M	L	R
F	L	H	M	R	R

Subsequently, E is inserted. Show the array after each step.

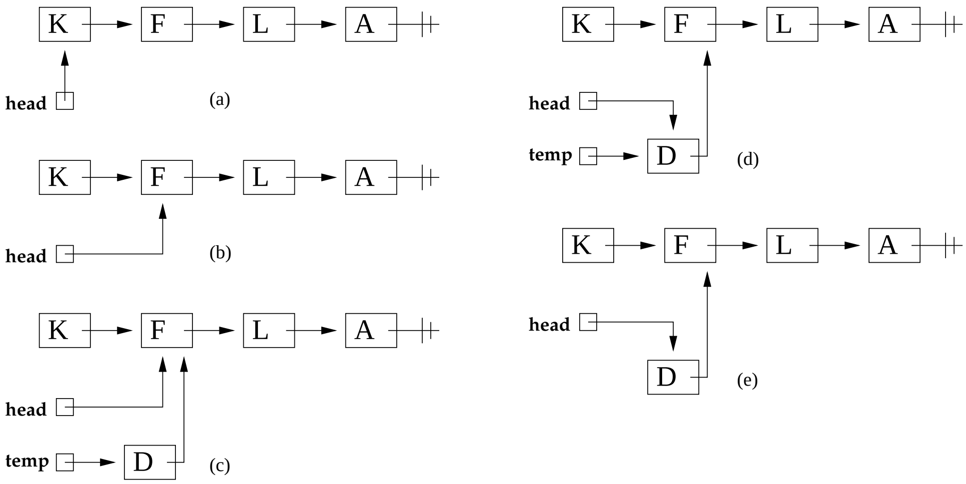
F	L	H	M	R	E
F	L	E	M	R	H
E	L	F	M	R	H

8. [20 points] Explain, using diagrams and text, but not pseudo-code, the linked list implementation of a stack. Represent the items on the stack by capital letters.

(a) Illustrate the stack with item K, F, L, A, in that order, where K is the top item.

(b) Starting with the previous stack, illustrate **pop.**

(c) Starting with the previous stack, illustrate **push,** where the item D is pushed onto the stack.



9. Let $G = (V, E)$ be a weighted directed graph with n vertices and m edges. The vertices are $v_1, v_2, \dots v_n$ and the edges are $e_1, e_2, \dots e_m$.

   Each e_j is a directed edge (x_j, y_j), where $x_j, y_j \in V$. $W[e_j] = W[x_j, y_j]$ is the given weight of the edge e_j. Note that a vertex could have several names. For example, If e_3 is a directed edge from v_2 to v_5, v_2 has the alternative name x_3 and v_5 has the alternative name y_3, and $W[e_3] = W[x_3, y_3] = W[v_2, v_5]$.

   Here is pseudo-code for the Bellman-Ford algorithm for the single source shortest path problem for G, where v_1 is the source. We will write $F[v_i]$ for the smallest weight of any path from v_1 to v_i that we have found so far, and we write $B[v_i]$ for the backpointer of that path. The arrays F and B are the outputs of the program.

   (a) [10 points] Fill in the missing line which assigns a backpointer.

   (b) [20 points] The main outer loop iterates $n - 1$ times. However, in most practical situations, The values of F are updated only during the first few iterations. Insert code (4 lines) to end the outer loop if there are no further changes to F. Assume that G has no negative cycle.

```
F[v_1] = 0;
for all i from 2 to n
   F[v_i] = ∞;
bool changed = true;
for all t from 1 to n−1
   if(changed)
     {
       changed = false;
       for all j from 1 to m
         if (F[x_j] + W[e_j] < F[y_j])
           {
             F[y_j] = F[x_j] + W[e_j];
             B[y_j] = x_j //assign a backpointer
             changed = true;
           }
     }
```