University of Nevada, Las Vegas Computer Science 477/677 Spring 2021

Answers to Assignment 2: Due Thursday February 4, 2021

Name:____

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1. Each of these code fragments takes if $O(n \log n)$.time, but not necessarily $\Theta(n \log n)$. Give the asymptotic complexity of each in terms of n, using Θ in each case.

   (a)
   ```
   for(int i = 1; i < n; i++)
    for(int j = 1; j < i; j = 2*j);
     cout << "Hello" << endl;
   ```
   $\int_{x=1}^{n} (lnx)dx = x\ln x - x|_{x=1}^{n} = \Theta(n \log n)$

   (b)
   ```
   for(int i = 1; i < n; i++)
    for(int j = i; j < n; j = 2*j);
     cout << "Hello" << endl;
   ```
   $\int_{x=1}^{n} (\ln n - \ln x)dx = x\ln x - x\ln x + x|_{x=1}^{n} = \Theta(n)$

   (c)
   ```
   for(int i = 1; i < n; i=2*i)
    for(int j = 1; j < i; j++);
     cout << "Hello" << endl;
   ```
   Let $\mathrm{k} = \log_2 \mathrm{i}$; then $2^{\mathrm{k}} = \mathrm{i}$.

   ```
   for(int k = 0; i < log_2 n; k++)
    for(int j = 1; j < 2^k; j++);
     cout << "Hello" << endl;
   ```

   Let x be the continuous analog of k and y the continuos analog of j.

   $$\int_{x=0}^{log_2 n} \int_{y=1}^{2^x} dydx = \int_{x=0}^{log_2 n} (2^x - 1)dx = \left.\frac{2^x - x}{\ln 2}\right|_0^{\log_2 n} = \frac{2^{\log_2 n} - 1}{\ln 2} = \frac{n-1}{\ln 2} = \Theta(n)$$

   (d)
   ```
   for(int i = 1; i < n; i=2*i)
    for(int j = i; j < n; j++);
      cout << "Hello" << endl;cd /home/larmore/Dropbox/Courses/CS477/S21
   ```

   Let $\mathrm{k} = \log_2 \mathrm{i}$; then $2^{\mathrm{k}} = \mathrm{i}$.

   ```
   for(int k = 0; i < log_2 n; k++)
    for(int j = 2^k; j < n; j++);
     cout << "Hello" << endl;
   ```

   Let x be the continuous analog of k and y the continuos analog of j.

   $$\int_{x=0}^{log_2 n} \int_{y=2^x}^{n} dydx = \int_{x=0}^{log_2 n} (n - 2^x)dx = \left.\left(nx - \frac{2^x}{\ln 2}\right)\right|_{x=0}^{\log_2 n}$$
   $$= n\log_2 n - \frac{2^{\log_2 n} - 1}{\ln 2} = n\log_2 n - \frac{n-1}{\ln 2} = \Theta(n \log n)$$

(e)
```
for(int i = n; i > 1; i=i/2)
 for(int j = i; j > 1; j--);
  cout << "Hello" << endl;
```

Same as (c). $\Theta(n)$

(f)
```
for(int i = n; i > 1; i=i/2)
 for(int j = n; j > i; j--);
  cout << "Hello" << endl;
```

Same as (d). $\Theta(n \log n)$

2. These problems are harder than the ones above. Given the asymptotic complexity of each fragment in terms of n, using Θ.

(g)
```
for(int i = 1; i < n; i=2*i)
 for(int j = 1; j < i; j=2*j);
  cout << "Hello" << endl;
```

Hint: Use substitution. Let `m = log n, k = log i, l = log j.`

```
for(int k = 0; k < m; k++)
 for(int l = 0; i < k; l++)
  cout << "Hello" << endl;
```

$\Theta(m^2) = \Theta(\log^2 n)$

(h)
```
for(int i = 2; i < n; i=i*i)
 cout << "Hello" << endl;
```

Hint: Use substitution. Let `m = log n, k = log i.`

Use the fact that $\log(x^y) = y \log x$

```
for(int k = 1; k < m; k=2*k)
 cout << "Hello" << endl;
```

$\Theta(\log m) = \Theta(\log \log n)$

(i)
```
for(int i = 2; i < n; i=i*i)
 for(int j = 1; j < i; j = 2*j)
  cout << "Hello" << endl;
```

Hint: Use substitution. Let `m = log n, k = log i, l = log j.`

```
for(int k = 1; k < m; k=2*k)
 for(int l = 0; l < k; l++)
```

$\Theta(m) = \Theta(\log n)$

(j)
```
for(int i = n; i > 1; i = log i)
 cout << "Hello" << endl;
```

Use the substitution $m = \log^* n, k = \log^* i$

```
for(int k = m; k > 0; k--)
```

Added on February 5:

The recusive definition of $\log^* x$ for any real number x is: $\log^* x = 0$ if $x \le 1$
$\log^* x = 1 + \log^*(\log x)$ if $x > 1$

Let i be the "old" value of i in the code, and $\bar{\imath}$ the "new" value of i, namely $\log i$. Let k be the old value of k and $\bar{k}$ the new value of k. Thus
$m = \log^* n$
$\bar{\imath} = \log i$
$k = \log^* i$
$\bar{k} = \log^* \bar{\imath}$
From the definition of $\log *$ we have:
$k = \log^* i = 1 + \log^* \log i = 1 + \log^* \bar{\imath} = 1 + \bar{k}$. Thus $\bar{k} = k - 1$, and the last parameter of the for statement is $k - -$.

End Added Text

The solution is $\Theta(m) = \Theta(\log^* n)$ where $\log^*$ is the *iterated logarithm.* For any positive real number x, $\log^* x$ is the number of times the logarithm function must be iteratively applied before the result is less than or equal to 1.

We use the base 2 logarithm. In that case, the iterated algorithm is sometimes written as $\lg^*$.

i. What is $\log^* 65536$? Answer: 4.

ii. What is $\log^* 65537$? Answer: 5.

iii. Let N be the number of baryons in the visible universe. (Neutrons and protons are baryons.) What is $\log^* N$? Answer: 5.

iv. It has been seriously conjectured that the radius of the entire universe is 10^{100} times the radius of the visible universe! If that is true, what is $\log^*$ of the number of baryons in the universe? Answer 5.

    $\log^*$ grows very slowly. However, it is not the slowest growing unbounded function that regularly arises in complexity theory. That honor goes to the inverse Ackermann function.

(k)
```
for(int i = 2; i < n; i = i*i)
 for(int j = 0; j < i; j++)
  cout << "Hello" << endl;
```

In my opinion, this is the hardest problem in this assignment. The time complexity of the code is O of one function of n and Ω of a different function of n, but is not Θ of any of the "usual" functions of n. Give both the O and the Ω answers, both of which are "usual" functions.[1]

Answer: The time complexity both $O(n)$ and $\Omega(\sqrt{n})$.

[1]By *usual functions* I mean the functions we have discussed so far in class, which include polynomials, logarithms, iterated logarithms, powers of logarithms, roots, and even the iterated logarithm $\log^*$.

The outer loop iterates $O(\log \log n)$ times. For each value of i used during the outer loop, , the inner loop iterates I times. Those values of i are numbers of the form 2^{2^k} for integers $k \geq 0$. That is,
$2^{2^0} = 2$,
$2^{2^1} = 2^2 = 4$,
$2^{2^2} = 4^2 = 16$,
$2^{2^3} = 16^2 = 256$,
$2^{2^4} = 256^2 = 65536$,
$2^{2^5} = 65536^2 = 4294967296$.
Since i increases rapidly, the time complexity of the code is dominated by the largest value of i generated in the outer loop, which is the largest value of 2^{2^k} less than n. Let's call that value I. For example, if $4 < n \leq 16$, $I = 4$; if $16 < n \leq 256$, $I = 16$; and if $256 < n \leq 65536$, $I = 256$; and so forth. Note that $I < n \leq I^2$, which implies that $\sqrt{n} \leq I < n$. The time complexity of the code is $\Theta(I)$, and we obtain our result.

3. Solve each of the following recurrences, giving the answer as Θ of a function of n.

(l) $F(n) = F(n/2) + n^2$
Master theorem: $A = 1$, $B = 2$, $C = 2$: Note that $A < B^C$.
Thus $F(n) = \Theta(n^C) = \Theta(n^2)$

(m) $F(n) = F(n/3) + 1$
Master theorem: $A = 1$, $B = 3$, $C = 0$: Note that $A = B^C$.
Thus $F(n) = \Theta(n^C \log n) = \Theta(\log n)$

(n) $F(n) = 16F(n/4) + n^2$
Master theorem: $A = 16$, $B = 4$, $C = 2$. Note that $A = B^C$.
Thus $F(n) = \Theta(n^C \log n) = \Theta(n^2 \log n)$

(o) $F(n) = F(n-1) + n^5$
Anti-derivative method: $\dfrac{F(n) - F(n-1)}{1} = n^5$
$F'(n) = \Theta(n^5)$
$F(n) = \Theta(n^6)$

(p) $F(n) = F(n - \log n) + \log n$
Anti-derivative method: $\dfrac{F(n) - F(n - \log n)}{\log n} = \dfrac{\log n}{\log n}$
$F'(n) = \Theta(1)$
$F(n) = \Theta(n)$

(q) $F(n) = 16F(n/4) + n$
Master theorem: $A = 16$, $B = 4$, $C = 1$. Note that $A > B^C$, and that $\log_B A = 2$.
Thus $F(n) = \Theta\big(n^{\log_B A}\big) = \Theta(n^2)$.