Lower Bounds on Numbers of Decisions

Theorem: At any point in a computation, if there are at most t decisions to be made, and all decisions are 2-way, then the number of possible outcomes is no greater that 2^t. Similarly, there all decisions are 3-way, then the number of possible outcomes is no greater that 3^t.

To put it another way, if there are N possible outcomes, and if all decisions are 2-way, then the number of decisions must be at least $\log_2 N$, while if all decisions are 3-way, the number of decisions must be at least $\log_3 N$.

Comparison Algorithms

For example, a comparison based sorting algorithm has only 2-way decisions. If the problem is to sort n items, the worst case is that the items are distinct, which implies that the number of possible outcomes is $n!$, since run of the algorithm must permute the input, and there are $n!$ permutations of n items.

A comparison based algorithm to sort 5 items must have at least 7 decisions, since $5! = 120$ and $\log_2 120$ is between 6 and 7. Can 5 items actually be sorted using at most 7 comparisons?

Suppose we need to find the median of 5 items, and also determine which two of the other four items are less than that median. There are 30 possible outcomes. (Do you see why?) Since $2^4 < 30 \leq 2^5$, the theorem says that we must use at least 5 comparisons in the worst case.

Can that problem be solved by a comparison based algorithm using just 5 comparisons in the worst case? We use this in our construction of the "median of medians" algorithm for finding the median of a set of n numbers with $O(n)$ comparisons.

Balance Scale Problems

You have a balance scale. When you put weights in both trays, the heavier weight side will go down, the other up. If the weights are the same, the two sides will stay at the same level. Thus, the scale is a device for making 3-way decisions.

Twelve Coin Problem

Suppose we have twelve coins, eleven of which are "good" and weigh the same, and one "bad," meaning it is it is either too heavy or too light. Can you determine, in three weighings, which coin is bad, and whether it is too heavy or too light?

The answer is yes.

Call the coins A, B, C, D, E, F, G, H, I, J, K, L.

1. Place A,B,C,D in the left tray, and E,F,G,H in the right tray. If they balance, go to step 2. If the left side goes down, go to step 3. If the right side goes down, go to step 5.

2. Place I,J,K in the left try, and A,B,C in the right tray. If they balance, the bad coin is L. Place L in

the left tray and A in the right. If the left side does down, L is too heavy, it it goes up, L is too light.

3. Either one of A,B,C,D is too heavy, or one of E,F,G,H is too light. I,J,K,L are good. Place A,B,E in the left tray, and C,F,I in the right. If it balances, go to step 4.

4. Place G in the left tray and H in the right. If it balances, D is too heavy. If the left side goes down, H is too light. If the right side goes down, G is too light.

5. Similar to step 2.

Thirteen Coin Problem

Suppose we have thirteen coins, twelve of which are "good" and weigh the same, and one "bad," meaning it is it is either too heavy or too light. Can you determine, in three weighings, which coin is bad, and whether it is too heavy or too light?

The answer is no. This surprising because there are 26 possible outcomes, which is less than $3^3 = 27$.

The first weighing must place an equal number of coins on each side. If that number is 4 or less and the scale balances, there are only two weighings left, which means the number of outcomes cannot exceed $3^2 = 9$. But any one of the remaining coins could be either heavy or light, which means there are at least 10 possible outcomes. So, we cannot finish.

On the other had, if the number in each try is 5 or 6, and if the left side goes down, one of the coins on the left could be heavy or one of the coins on the right could be light. That makes at least 10 possible outcomes, but there are only two remaining weighings. So, we cannot finish.

Fourteen Coin Problem

Suppose we have thirteen coins, twelve of which are "good" and weigh the same, and one "bad," meaning it is it is either too heavy or too light. But we are also given a fourteenth coin which we know is "good." Can you determine, in three weighings, which coin is bad, and whether it is too heavy or too light?

The answer is yes.

Place the coin which is known to be good in the left tray, along with four of the "suspect" coins. Place five suspect coins in the right tray. There will be four suspect coins not used in this weighing.

I used our theorem to guide my choice of the first step. Can you finish this algorithm? Keep the theorem in mind.