

Longest Monotone Subsequence

Problem: Given a sequence $\sigma = x[1], \dots, x[n]$, find the longest strictly monotone increasing subsequence of σ . (The Greek letter sigma.) To simplify our notation, we write “monotone” to mean strictly monotone increasing.

$\tau[t]$, the sequence $x[1], \dots, x[t]$. (The Greek letter tau.) Note that $\sigma = \tau[n]$.

$v[t]$, the longest monotone subsequence of $\tau[t]$. Defaults to 0. (The Greek letter upsilon, or ypsilon.)

$U[t]$, the length of $v[t]$

$j[t]$ the index of the last term of $v[t]$

$\varrho[t]$, the longest monotone subsequence of $\tau[t]$ ending at $x[t]$. (The Greek letter rho.)

$R[t]$, the length of $\varrho[t]$. Defaults to 0.

$p[t]$, a back pointer, the index of the second-to-last term of $\varrho[t]$. Defaults to 0.

Here is the code:

```
 $x[0] = -\infty$ 
```

```
 $U[0] = 0$ 
```

```
 $R[0] = 0$ 
```

```
For all  $t$  from 1 to  $n$ 
```

```
{
```

```
  Pick  $0 \leq i < t$  such that  $R[i]$  is maximum subject to  $x[i] < x[t]$ 
```

```
   $R[t] = R[i] + 1$ 
```

```
   $p[t] = i$ 
```

```
}
```

```
if( $R[t] > U[t - 1]$ )
```

```
{
```

```
   $U[t] = R[t]$ 
```

```
   $j[t] = t$ 
```

```
}
```

```
else
```

```
{
```

```
   $U[t] = U[t - 1]$ 
```

```
   $j[t] = j[t - 1]$ 
```

```
}
```

The longest monotone subsequence is found by following the back pointers starting at index $j[n]$, and has length $U[n]$. Line 6 of the code (starting with the word “Pick”) dominates the time complexity. If the search is done linearly, the time complexity of the algorithm is $O(n^2)$. However, i can be found in $O(\log n)$ time using binary search, yielding time complexity $O(n \log n)$.