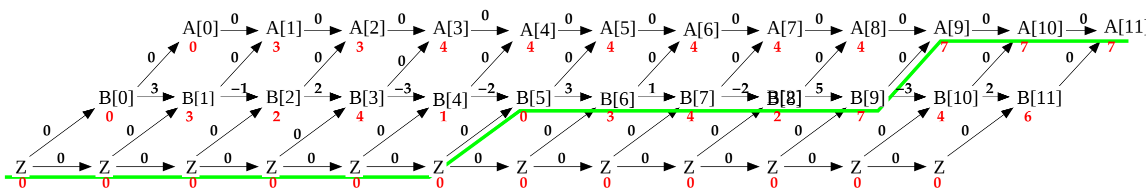
Maximum Contiguous Subsequence

Recall problem 6.1 on page 177 of your textbook. Let $a_1, \ldots a_n$ be the sequence, which may include both positive and negative numbers. The problem is to find the maximum sum of any contiguous subsequence.

Let C_k be the maximum of zero, and the maximum sum of any contiguous subsequence of $a_1, \ldots a_k$, and let B_k be the maximum of zero, and the maximum sum of any contiguous subsequence that ends at a_k, We let $B_0 = C_0 = 0$ by default.

```
B[0] = 0;
C[0] = 0;
for(int k = 1; k <= n; k++)
 {
  B[k] = max(0,B[k-1]+a[k]);
  C[k] = max(C[k-1],B[k]);
 }
cout << ‘‘The maximum sum of any ":
cout << "contiguous subsequence is ";
cout << C[n] << endl;
```

The problem can be reduced to the maximum path problem in a weighted layered directed graph, as shown below. In our reduction, each layer has up to three nodes, $A[k-1]$, $B[k]$, and Z, which is always zero.



We illustrate the layered graph obtained from the input sequence $3, -1, 2, -3, -2, 3, 1, -2, 5, -3, 2$ of length $n = 11$. The red numerals are the solution to the single source maximum path problem, and the green line indicates the maximum path to A_n, which shows that the solution is the contiguous subsequence $3, 1, -2, 5$, whose total is 7.

Parallel Computation

Instead of using dynamic programming, we can solve the maximum path problem by $(\max, +)$ matrix multiplication. Each layer is represented by a vector of length 1, 2, or 3. The first of those vectors is $(\ 0\ )$. Each subsequent vector is is the $(\min, +)$ matrix product of the previous vector by a matrix. The product of these matrices can be computed in parallel by n processors in $O(\log n)$ time. In the example we have the product of thirteen matrices:

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -1 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & 2 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -3 & -\infty \\ -\infty & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -2 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & 3 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & 1 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -2 & -\infty \\ -\infty & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\infty & -\infty \\ 0 & 5 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -3 & -\infty \\ -\infty & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\infty \\ 0 & 2 \\ -\infty & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \end{pmatrix}$$