Quicksort

**QUICKSORT**($A, p, r$)

```plaintext
    if $p < r$
    then $q \leftarrow \text{PARTITION}(A, p, r)$
    QUICKSORT($A, p, q - 1$)
    QUICKSORT($A, q + 1, r$)
```

**PARTITION**($A, p, r$)

```plaintext
    $x \leftarrow A[r]$
    $i \leftarrow p - 1$
    for $j \leftarrow p$ to $r - 1$
      do if $A[j] \leq x$
        then $i \leftarrow i + 1$
        exchange $A[i] \leftrightarrow A[j]$
    exchange $A[i + 1] \leftrightarrow A[r]$
    return $i + 1$
```
Performance of **QuickSort**

Let $T(n) = \text{number of } A[j] \leq x \text{ comparisons}$.

Worst-case: **PARTITION** always returns $p$ or $r$. $T(n) = T(n-1) + T(0) + n - 1$ implies $T(n)$ is $\Theta(n^2)$.

Proof: $T(0) = 0$, so recurrence simplifies to:

$$T(n) = T(n-1) + n - 1 = \sum_{k=1}^{n} (k - 1)$$

Here is an upper bound on $T(n)$.

$$T(n) = \sum_{k=1}^{n} (k - 1) \leq n(n - 1) < n^2$$

Here is a lower bound on $T(n)$.

$$T(n) = \sum_{k=1}^{n} (k - 1) \geq \sum_{k=\left\lfloor n/2 \right\rfloor}^{n} (k - 1) \geq (n/2)(n/2 - 1) = n^2/4 - n/2$$
Best-case: \texttt{PARTITION} returns \([ (p + r) / 2 ] \).

By Master Method, \( T(n) = 2T(n/2) + n - 1 \) implies \( T(n) \in \Theta(n \lg n) \).

To understand average-case, consider average partition size.

Average partition: Let \( s_1 \) and \( s_2 = \) region sizes
Assume that \( s_1 \) is equally likely to be any real number from 0 to \( n \), and that \( s_1 + s_2 = n \)

With prob. 1/4, \( s_1 \leq n/4 \), and \( s_2 \geq 3n/4 \).
With prob. 1/4, \( n/4 \leq s_1 \leq n/2 \leq s_2 \leq 3n/4 \).
With prob. 1/4, \( 3n/4 \geq s_1 \geq n/2 \geq s_2 \geq n/4 \).
With prob. 1/4, \( s_1 \geq 3n/4 \), and \( s_2 \leq n/4 \).

So with prob. 1/2, \( \max(s_1, s_2) \leq 3n/4 \).
and with prob. 1/2, \( \max(s_1, s_2) \geq 3n/4 \).

This implies a median value of about \( 3n/4 \) for the maximum of \( s_1 \) and \( s_2 \). When does median = average?
Average-Case: Each partition equally likely.

This can be guaranteed by adding the following randomization to \texttt{Partition}:

\begin{verbatim}
\textbf{Randomized-Partition}(A, p, r)
    i ← \texttt{Random}(p, r)
\end{verbatim}

▷ Rest of procedure same as \texttt{Partition}

This guarantees that every element is equally likely to be the pivot.

\[
T(n) = n - 1 + \frac{1}{n} \sum_{k=0}^{n-1} (T(k) + T(n - 1 - k))
\]

\[
= n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} T(k)
\]

\[
T(n - 1) = n - 2 + \frac{2}{n - 1} \sum_{k=0}^{n-2} T(k)
\]

\[
nT(n) - (n - 1)T(n - 1)
    = 2(n - 1) + 2T(n - 1)
\]
\[ nT(n) = 2(n - 1) + (n + 1)T(n - 1) \]

\[
\frac{T(n)}{n + 1} < \frac{2}{n} + \frac{T(n - 1)}{n}
\]

\[
< \frac{2}{n} + \frac{2}{n - 1} + \frac{T(n - 2)}{n - 1}
\]

\[
< 2 \sum_{k=2}^{n} \frac{1}{k} < 2 \ln n
\]