Roots of Unity

Polar Representation of Complex Numbers

For $z \in \mathbb{C}$, $\arg z$ is defined to be the radian measure of the angle from the line $[0, 1]$ to the line $[0, z]$, provided $z \neq 0$. The indeterminacy of $\arg z$ is integral multiples of 2π. For example, $\arg(-i) = \frac{3\pi}{2}$, but is also equal to $-\frac{\pi}{2}$. It is common to use the variable θ for $\arg z$. The polar representation of z is the ordered pair (r, θ) where $r = |z|$ and $\theta = \arg z$.

1. If $\arg z = \theta$, than $\arg z$ is also equal to $\theta \pm 2\pi$.
2. If the polar representation of z is (r, θ), the "$a + bi$" representation of z is $r(\cos\theta + i \sin\theta)$.
3. If the polar representations of z_1 and z_2 are (r_1, θ_1) and (r_2, θ_2), respectively, the polar representation of $z_1 z_2$ is $(r_1 r_2, \theta_1 + \theta_2)$.

The Exponential Function

Using the Taylor series, for any complex number z, e^z is a complex number. If z is an imaginary number, that is, a real multiple of i, the square root of -1, e^z is on the unit circle in the Gaussian plane. If x is a real number, we can see from the Taylor series of the three functions, that $e^{ix} = \cos(x) + i \sin(x)$. Thus $e^{i\pi} = -1$.

Exponential of an Imaginary

Theorem 1 *If θ is any real number, $e^{\theta i} = \cos\theta + i \sin\theta$.*

We prove Theorem 1. using Taylor series. Note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i \ldots$.

$$\begin{aligned}
e^{\theta i} &= 1 + \theta i + \frac{(\theta i)^2}{2!} + \frac{(\theta i)^3}{3!} + \frac{(\theta i)^4}{4!} + \frac{(\theta i)^5}{5!} + \frac{(\theta i)^6}{6!} + \frac{(\theta i)^7}{7!} + \cdots \\
&= 1 + \theta i + \frac{\theta^2}{2!} i^2 + \frac{\theta^3}{3!} i^3 + \frac{\theta^4}{4!} i^4 + \frac{\theta^5}{5!} i^5 + \frac{\theta^6}{6!} i^6 + \frac{\theta^7}{7!} i^7 + \cdots \\
&= 1 + \theta i - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} i + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} i - \frac{\theta^6}{6!} - \frac{\theta^7}{7!} i + \cdots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\
&= \cos\theta + i \sin\theta
\end{aligned}$$

Note that $\left|e^{\theta i}\right| = 1$ and $\arg e^{\theta i} = \theta$.

Theorem 2 *For $z = a + bi \in \mathcal{C}$, $|e^z| = e^a$ and $\arg e^z = b$.*

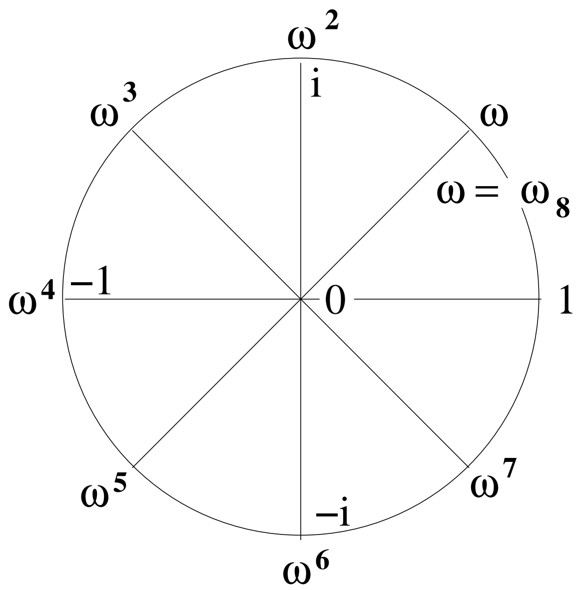

Figure 1: 8^{th} Roots of Unity

Roots of Unity

For any positive integer n, every complex number except 0 has n distinct n^{th} roots. The nth roots of unity, that is roots of 1, are all $e^{\frac{2k\pi}{n}}$ for $k = 0, 1, \ldots n-1$. Let $\omega_n = e^{\frac{2\pi}{n}}$, the principal n^{th} root of unity. For the FFT,

Computing Roots of Unity

1. $w_2 = -1$
2. $\overline{\omega_n^k} = \omega_n^{n-k}$
3. $\omega_{2n}^2 = \omega_n$, and the two square roots of ω_n^k are ω_{2n}^k and ω_{2n}^{k+n}
4. $\omega_{2n}^k + \omega_{2n}^{k+n} = 0$
5. $\omega_{2n}^k * \omega_{2n}^{k+n} = -\omega_n^k = \omega_n^{k+\frac{n}{2}}$

If n is a power of 2, the real and imaginary parts of w_n can be expressed using fractions and square roots. Using rule 3 above, we obtain

$w_4 = i$

$$w_8 = \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}\, i = \frac{1+i}{\sqrt{2}}$$

$$w_{16} = \sqrt{\frac{1+\sqrt{\frac{1}{2}}}{2}} + \sqrt{\frac{1-\sqrt{\frac{1}{2}}}{2}}\, i$$