Extra Difficulty Assignment: Rice Numbers

Recall that languages are equivalent to 0/1 problems, and we'll use those terms interchangeably.

We use the following notation.

$\mathbb{N}$ = the natural numbers (the positive integers).
$\mathbb{Z}$ = the integers.
$\mathbb{Q}$ = the rational numbers.
$\mathbb{R}$ = the real numbers.
$\mathbb{I}$ = the unit interval = $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$
A *fraction* is a string of the form "¡n¿/¡m¿" where $n, m \in \mathbb{Z}$ and $m \neq 0$. The *value* of that fraction is n/m.

1. What is a *diadic* rational number?
2. For any alphabet Σ, every language over Σ has a *Rice number* in the unit inteval. (The alphabet must be specified.) Give a definition of the Rice number of a language. (Henceforth, we assume that all languages are over Σ.)
3. What is the Rice number of Σ^*? What is the Rice number of the empty language?
4. Prove that a language is recursive (decidable) if and only if its Rice number is a recursive real number.
5. Is it possible for two different languages to have the same Rice number? Hint: the word "diadic" should appear in your explanation.
6. Prove that if $x \in \mathbb{I}$ then x is the Rice number of some language.
7. Suppose x and y are the Rice numbers of languages X and Y, and that Y is the complement of X, that is, $X \cup Y = \Sigma^*$ and $X \cap Y = \emptyset$. Prove that $x + y = 1$.
8. We define a sequence $\sigma = x_1, x_2, \ldots$ of strings to be *recursive* if there is some machine which writes σ. (If the sequence is infinite, the machine will never halt.)
9. Prove that there is some recursive sequence of fractions whose sequence of values converges to a real number which is not recursive.