CS 456/656 Third Examination November 20, 2024

The entire examination is 330 points.

1. True/False/Open

   (i) [5 points] **T** Every context-free language is $\mathcal{NC}$.

   (ii) [5 points] **T** Every $\mathcal{P}$–TIME problem is $\mathcal{NC}$-reducible to the Boolean Circuit problem.

   (iii) [5 points] **T** If x is a recursive real number, the set of fractions whose values are less than x is a decidable language.

   (iv) [5 points] **T** The ancient Greek mathematicians believed that all real numbers were rational, until one of them proved that irrational numbers exist.

   (v) [5 points] **T** There are infinitely many prime integers.

   (vi) [5 points] **O** $\mathcal{NP} = \mathcal{P}$–SPACE.

   (vii) [5 points] **O** $\mathcal{NC} = \mathcal{P}$–TIME.

   (viii) [5 points] **T** Matrix multiplication is $\mathcal{NC}$.

   (ix) [5 points] **T** The jigsaw problem is $\mathcal{NP}$–complete.

   (x) [5 points] **O** The furniture mover's problem is $\mathcal{NP}$–complete.

   (xi) [5 points] **T** Every context-free language is generated by some CNF (Chomsky Normal Form) grammar.

   (xii) [5 points] **F** $\mathcal{P}$–TIME = EXP–TIME.

   (xiii) [5 points] **O** The Boolean Circuit problem is inherently sequential.

   (xiv) [5 points] **F** Every decidable language is context-sensitive.

   (xv) [5 points] **T** Every context-sensitive language is decidable.

   (xvi) [5 points] **F** Given any uncountable set S, there is a 1-1 correspndence between S and the set of real numbers.

   (xvii) [5 points] **F** The asymptotic time complexity of a language (problem) is never greater than its asymptotic space complexity.

   (xviii) [5 points] **T** The halting problem is accepted by some machine.

   (xix) [5 points] **T** The set of all strings of the form $\langle G_1\rangle\langle G_2\rangle$ such that G_1 and G_2 are context-free grammars which are **not** equivalent is recursively enumerable.

   (xx) [5 points] **F** If a regular language is accepted by an NFA with n states, it is accepted by a DFA with n states.

2. [20 points] What does it mean to say that a set X is countable?

There is a 1-1 correspondence betwee X and the natural numbers.

3. [20 points] State the pumping lemma for context-free languages.

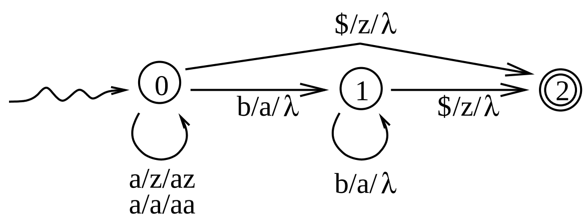
For any context-free language L
There exists a positive number p such that
For any $w \in L$ of length at least p
There exist strings u, v, x, y, z such that the following hold:

1. $w = uvxyz$
2. $|vxy| \leq p$
3. v and y are not both empty
4. For any $i \geq 0$, $uv^i xy^i z \in L$

4. [20 points] Give an example of a language L which is not CF, but whose complement is context-free.

$\{a^n b^n c^n : n \geq 0\}$

5. [20 points] Sketch a DPDA that accepts the language $\{a^n b^n : n \geq 0\}$

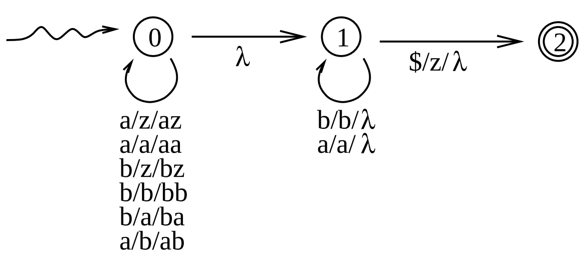


6. [10 points] Give a context-sensitive language that is not context-free.

$\{a^n b^n c^n : n \geq 0\}$

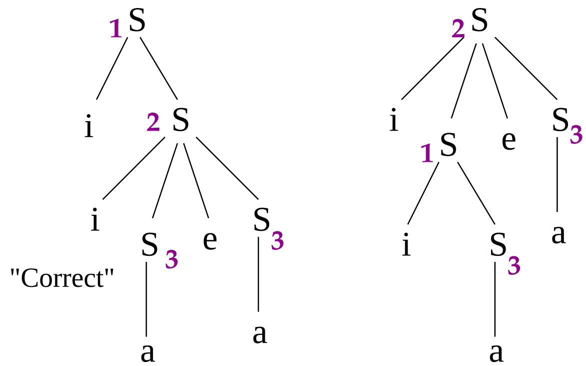
7. [20 points] Sketch a PDA (not a DPDA) which accepts the palindromic language generated by the following CF grammar:

1. $S \to aSa$
2. $S \to bSb$
3. $S \to \lambda$



8. [20 points] Consider the context-free grammar G below, where S is the only variable. Prove that G is ambiguous by given two different parse trees for the string *iiaea* using G. Label the internal nodes of each parse tree with production numbers.

1. $S \to iS$
2. $S \to iSeS$
3. $S \to a$



9. [20 points] Prove that $\sqrt{2}$ is irrational.

By contradiction. Assume $\sqrt{2}$ is rational. Then it is the value of a fraction $\frac{p}{q}$, where p and q have no common divisor greater than 1.

$$\begin{aligned}
\sqrt{2} &= \frac{p}{q} \\
2 &= \frac{p^2}{q^2} \\
2q^2 &= p^2 \text{ hence } p^2 \text{ is even, hence } p \text{ is even} \\
p &= 2k \text{ for some integer } k \\
p^2 &= 4k^2 \\
2q^2 &= 4k^2 \\
q^2 &= 2k^2 \text{ hence } q^2 \text{ is even, hence } q \text{ is even}
\end{aligned}$$

p and q have the common divisor 2, contradiction. We conclude that $\sqrt{2}$ is irrational.

10. [20 points] Prove that every language which is enumerated in canonical order by some machine is decidable.

The following program decides L. Let $w_1, w_2, \dots$ be the canonical order of L, which we can use as an input to our program.

Read w
For all w_i in canonical order
If $w = w_i$ halt and accept
If $w < w_i$ (in the canonical order) halt and reject

11. [20 points] Give a polynomial time reduction from the subset sum problem to the partition problem. The reduction maps an instance $(x_1, x_2, \dots x_n, K)$ of the subset sum problem to the instance $(x_1, x_2, \dots x_n, K+1, S-K+1)$ of the partition problem, where $S = \sum_{i=1}^{n} x_i$.

12. [20 points] Prove that HALT is undecidable.

By contradiction. Assume HALT is decidable. Let P be the following program

Read a machine description $\langle M \rangle$.
If M halts with input $\langle M \rangle$, enter an infinite loop.
Else halt.

The input to P could be $\langle P \rangle$. If so, does it halt?

If P halts with input $\langle P \rangle$, then, with input $\langle P \rangle$, the code will cause P to enter an infinite loop, that is, not halt, contradiction.

Alternatively, if P does not halt with input $\langle P \rangle$, then, with intput $\langle P \rangle$, the code will cause P to halt, contradiction.

In either case, we have a contradiction. We conclude that HALT is not decidable.

13. [20 points] Prove that every recursively enumerable language is accepted by some machine.

Let $w_1, w_2 \ldots$ be an enumeration of L written by some machine. The following program accepts L.

Read w
For i from 1 to ∞
If $w = w_i$ halt and accept

14. This problem is worth more than just points.

Prove that 2-SAT is $\mathcal{P}$–TIME. I never proved this in class, so it's a real challenge. Can you do it?

Don't try it until you finish all other problems on the exam!