Regular Languages are $\mathcal{NC}$

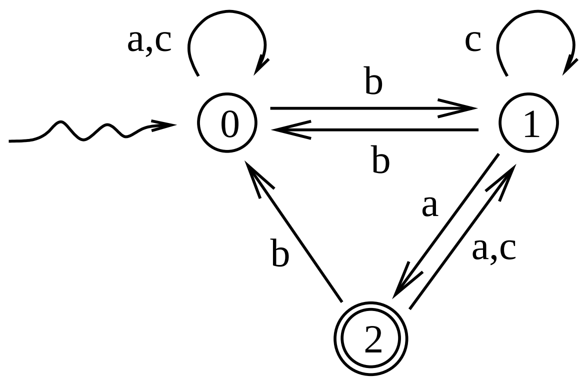
Let L be a regular language, and let M be a DFA which accepts (actually, decides) L. Using M, we design an $\mathcal{NC}$ algorithm which decides L in $O(\log n)$ time using $O(n)$ processors, where n is the length of the input string w.

$M = (Q, \Sigma, \delta, q_0, F)$. Recall Q is the set of states of M, Σ is the input alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, $q_0 \in Q$ is the start state, and $F \subseteq Q$ is the set of final states. We extend the transition function to $\delta^* : Q \times \Sigma^* \to Q$ inductively: $\delta^*(q, \lambda) = q$, and $\delta^*(q, aw) = \delta^*(\delta(q, a), w)$ for any $a \in \Sigma$, $q \in Q$. If $w \in \Sigma^*$, then $w \in L$, *i.e.,* is accepted by M, if $\delta^*(q_0, w) \in F$. Equivalently, we describe the transition function of M by a function $\delta^*(\ , x) : Q \to Q$. for any $x \in \Sigma^*$; where $\delta^*(\ , x)(q) = \delta^*(q, x)$ for all $q \in Q$.

We now describe an $\mathcal{NC}$ algorithm $\mathcal{A}$, which decides whether a given string is a member of L. To simplify our construction, we assume that the length of the input string is a power of 2, although it is a simple matter to generalize to arbitrary n: augument Σ with a special "do nothing" symbol $\bullet$, which we call a blank. Define $\delta(q, \bullet) = q$ for any $q \in Q$. Let w^* be the string obtained by padding the input string w with just enough blanks to bring its length to a power of 2. For example, if $w = aabcacbabbcaa$ we let $w^* = aabcacbabccca\bullet\bullet\bullet$. Let $n = 2^m = |w^*|$. Let $\mathfrak{S}$ be the set of consisting of all subintervals obtained by breaking w^* into 2^i pieces each of length 2^{m-i}, for all $0 \le i \le m$. Thus $\mathfrak{S}$ consists of all subintervals of length 1, $n/2$ subintervals of length 2, $n/4$ subintervals of length 4, and so forth; these will include 2 subintervals of length $n/2$ and one of length n, namely w itself. The cardinality of $\mathfrak{S}$ is $2n - 1$. Each member of $\mathfrak{S}$ of length 2^i, for $i > 0$, is the concatenation of two members of $\mathfrak{S}$ of length 2^{i-1}. We let $u_{i,j}$ be the j^{th} member of $\mathfrak{S}$ of length 2^i, for $0 \le i \le m$ and $1 \le j \le 2^{n-i}$. That is, $u_{i,j}$ is the substring of w^* of length 2^i ending at the $(2^i j)^{\text{th}}$ place of w^*. $\mathcal{A}$ has $1 + m$ *phases*, which we number $0, 1, \ldots m$. Phase i of $\mathcal{A}$ computes $\delta^*(\ , u_{i,j})$ for all $1 \le j \le 2^i$, takes $O(1)$ time and uses 2^{m-i} processors. The functions $\delta^*(\ , u_{0,j})$ for all j can simply be read off the state diagram of M. For $i > 0$, $\delta^*(\ , u_{i,j})$ is simply the composition of the functions $\delta^*(\ , u_{i-1,2j-1})$ and $\delta^*(\ , u_{i-1,2j})$, for all $1 \le j \le 2^{m-i}$. For example, in Phase 1 of the example computation below, $\delta^*(\ , bc)$ is the composition of $\delta^*(\ , b)$ with $\delta^*(\ , c)$

Example

Let M be given by the state diagram below. For simplicitly, we dispense with the clumsy "q_i" notation and write simply i. Thus $Q = \{0, 1, 2\}$, the start state is 0, and $F = \{2\}$.



Let $w = aabcacbabccca$. The sequential computation of M with input w takes 13 steps. Since $2 \in F$, w is accepted.

$$0 \xrightarrow{a} 0 \xrightarrow{a} 0 \xrightarrow{b} 1 \xrightarrow{c} 1 \xrightarrow{a} 2 \xrightarrow{c} 1 \xrightarrow{b} 0 \xrightarrow{a} 0 \xrightarrow{b} 1 \xrightarrow{c} 1 \xrightarrow{c} 1 \xrightarrow{c} 1 \xrightarrow{a} 2$$

Padding with blanks to obtain a length of 16, a power of 2, we let $w^* = aabcacbabccca\bullet\bullet\bullet$. Execute $\mathcal{A}$ in five phases using 16 processors.

Phase 0:

a	a	b	c	a	c	b	a	b	c	c	c	a	•	•	•
$0 \to 0$	$0 \to 0$	$0 \to 1$	$0 \to 0$	$0 \to 0$	$0 \to 0$	$0 \to 1$	$0 \to 0$	$0 \to 1$	$0 \to 0$	$0 \to 0$	$0 \to 0$	$0 \to 0$	$0 \to 0$	$0 \to 0$	$0 \to 0$
$1 \to 2$	$1 \to 2$	$1 \to 0$	$1 \to 1$	$1 \to 2$	$1 \to 1$	$1 \to 0$	$1 \to 2$	$1 \to 0$	$1 \to 1$	$1 \to 1$	$1 \to 1$	$1 \to 2$	$1 \to 1$	$1 \to 1$	$1 \to 1$
$2 \to 1$	$2 \to 1$	$2 \to 0$	$2 \to 1$	$2 \to 1$	$2 \to 1$	$2 \to 0$	$2 \to 1$	$2 \to 0$	$2 \to 1$	$2 \to 1$	$2 \to 1$	$2 \to 1$	$2 \to 2$	$2 \to 2$	$2 \to 2$

Phase 1:

aa	bc	ac	ba	bc	cc	a•	••
$0 \to 0$	$0 \to 1$	$0 \to 0$	$0 \to 2$	$0 \to 1$	$0 \to 0$	$0 \to 0$	$0 \to 0$
$1 \to 1$	$1 \to 0$	$1 \to 1$	$1 \to 0$	$1 \to 0$	$1 \to 1$	$1 \to 2$	$1 \to 1$
$2 \to 2$	$2 \to 0$	$2 \to 1$	$2 \to 0$	$2 \to 0$	$2 \to 1$	$2 \to 1$	$2 \to 2$

Phase 2:

aabc	acba	bccc	a•••
$0 \to 1$	$0 \to 2$	$0 \to 1$	$0 \to 0$
$1 \to 0$	$1 \to 0$	$1 \to 0$	$1 \to 2$
$2 \to 0$	$2 \to 0$	$2 \to 0$	$2 \to 1$

Phase 3:

aabcacba	bccca•••
$0 \to 0$	$0 \to 2$
$1 \to 2$	$1 \to 0$
$2 \to 2$	$2 \to 0$

Phase 4:

aabcacbabccca•••
$0 \to 2$
$1 \to 0$
$2 \to 0$

The computation at Phase 4 tells us that $\delta^*(0, aabcacbabccca\bullet\bullet\bullet) = 2$, a final state. Thus $aabcacbabccca \in L$.