The Partition Problem is $\mathcal{NP}$–COMPLETE

We define an instance of the *partion problem* to consist of any finite sequence of positive numbers. Y is a member of the language *Partition* if and only if the terms of Y can be partitioned into two subsequences, each of total half the total of the terms of Y.

Partition is clearly $\mathcal{NP}$, since, if $Y \in$ Partition, either one of the subsequences is a polynomial length certificate.

Reduction of Subset_Sum to Partition

We define an instance of the *subset sum problem* to be an ordered pairs (X, C) where C is a number and X is a sequence of positive numbers. The ordered pair (C, X) is a member of the language *Subset_Sum* if there is a subsequence of X whose sum is C.

We define R, a $\mathcal{P}$–TIME reduction of Subset_Sum to Partition. Let (X, C) be an instance of the subset sum problem, where $X = x_1, \ldots x_n$. Let $S = \sum_{i=1}^{n} x_i$. Without loss of generality, $0 \leq C \leq S$.

Let $R(X, C)$ be a sequence $Y = y_1, \ldots y_n, y_{n+1}, y_{n+2}$, where $y_i = x_i$ for $i \leq n$, $y_{n+1} = C + 1$, and $y_{n+2} = S - C + 1$. Then $Y \in$ Partition if there are two disjoint subsequences of Y each of sum $S + 1$.

Lemma 1 *$(X, C) \in$ Subset_Sum if and only if $Y \in$ Partition.*

Proof: Suppose $(X, C) \in$ Subset_Sum. Let Z be a subsequence of X whose sum is C. Then $Z + \{y_{n+2}\}$ is a subseqence of Y whose sum is $S+1$. Conversely, suppose Y is partitioned into disjoint subsequences each of sum $S+1$. Neither of those subsequences contains both of the last two terms of Y, since their total exceeds $S + 1$. Thus one subsequence, say W, contains $y_{n+2} = S - C + 1$. Remove y_{n+2} from W to obtain a subsequence of X of whose sum is C. ∎

We immediately have:

Theorem 1 *If Subset_Sum is $\mathcal{NP}$–COMPLETE then Partition is $\mathcal{NP}$–COMPLETE*