Matrix Multiplication

Matrix multiplication is defined if all entries of each matrix lie in a domain with two operations, $\oplus$ and $\otimes$, where $\oplus$ is commutative and associative, $\otimes$ is associative, and $\otimes$ distributes over $\otimes$ on both left and right.

Given an $n \times m$ matrix A and an $m \times p$ matrix B, the product AB is an $n \times p$ matrix where $AB[i,k] = \bigoplus_{j=1}^{m} A[i,j] \otimes B[j,k]$. Matrix multiplication is associative: for matrices A, B, C such that AB and BC are defined: $(AB)C = A(BC)$, thus we can simply write ABC. If A is a square matrix, that is, $n \times n$, we write $A^2 = AA$, $A^3 = AAA$, and so forth.

- Convential matrix multiplication uses addition for $\oplus$ and multiplication for $\otimes$.
- Boolean matrix multiplication requires entries to be either 0 (false) or 1 (true) and use disjunction for $\oplus$ and conjunction for $\otimes$.
- Tropical matrix multiplication uses minimum for $\oplus$ and addition for $\otimes$.

Matrix Multiplication is $\mathcal{NC}$

In general, matrix multiplication is in Nick's Class, meaning that it can be computed in polylogarithmic time using polynomially many processors working in parallel.

Theorem 1 *If each $\oplus$ and each $\otimes$ operation takes constant time, the product of two $n \times n$ matrices can be computed in $O(\log n)$ time by $n^3/\log n$ parallel processors.*

Application to the All Pairs Minpath Problem

Let G be a weighted digraph with n vertices. Let M_G be the $n \times n$ matrix representing G, that is, $M_G[i,j] = W(i,j)$, the weight of the arc from vertex i to vertex j. We allow an entry to be ∞ if there is no arc.

Theorem 2 *Using tropical matrix multiplication, $M_G^{n-1}[i,j]$ is the least weight of any directed path in G from vertex i to vertex j.*

Thus, using tropical matrix multiplication, we can compute the least weight of any path between all pairs of vertices in a directed graph of size n in $O(\log^2 n)$ time using $n^3/\log n$ parallel processors.