CSC 456/656 Fall 2025 Answers to Study for Third Examination April 9, 2025

1. Review **tf1ans.pdf**, **tf2ans.pdf** and **tf3ans.pdf**.

2. Use the pumping lemma for regular languages to prove that $L = \{a^n b^n\}$ is not regular.

   The proof is already posted.

3. Use the pumping lemma for context-free languages to prove that $L = \{a^n b^n c^n\}$ is not context-free.

   Assume that L is context-free. Let p be the pumping length of L. Let $w = a^p b^p c^p$, which has length at least p. Then there must exist strings u, v, x, y, z such that:
   1. $w = uvxyz$
   2. $|vxy| \leq p$
   3. $|v| + |y| \geq 1$
   4. For any integer $i \geq 0$, $uv^i xy^i z \in L$.

   The length of the substring b^p of w is greater than the length of vxy, hence vxy either contains no a or no c.

   Case 1. vxy contains on a. Let $i = 0$. By 4., $uxz \in L$. Since vy cannot be empty, $|uxz| < |w| = 3p$. Thus Since all a's are in uxz, its length must be $3p$, contradiction. We conclude that L is not context-free.

4. Prove that the complement of $L = \{a^n b^n c^n\}$ is context-free. Hint: A CF grammar for that language has lots of productions.

   The complement of L consists of all strings over $\Sigma = \{a, b, c\}$ which can be proved to not be in L for some reason.

   (i) Let L_1 consist of all strings over Σ which contain the substring ba.

   (ii) Let L_2 consist of all strings over Σ which contain the substring ca.

   (iii) Let L_3 consist of all strings over Σ which contain the substring cb.

   (iv) Let L_4 consist of all strings of the form $a^i b^j c^k$ such that $i < j$.

   (v) Let L_5 consist of all strings of the form $a^i b^j c^k$ such that $j < i$.

   (vi) Let L_6 consist of all strings of the form $a^i b^j c^k$ such that $j < k$.

   (vii) Let L_7 consist of all strings of the form $a^i b^j c^k$ such that $k < j$.

   Clearly, $L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L7$ is the complement of L. We only need to show that each L_i is context-frees.

   L_1 is regular, since it has the regular expression $(a + b + c)^* ba(a + b + c)^*$. Similarly, L_2 and L_3 are also regular. Thus, those three languages are all CF. L_4 is generated by the CFG
   $S \to AaC$
   $A \to a$
   $A \to aAb$
   $A \to \lambda$

$C \to cC$
$C \to \lambda$.

Thus L_4 is context-free. Similarly, L_5, L_6 and L_7 are also context-free. The union of finitely many context-free languages is context free, and we are done.

5. Prove that the set of recursive real numbers is countable.

   Every recursive real number x has a unique recursive function d such that $d(n)$ is the n^{th} decidmal digit of x after the decimal point. There are only countably many such functions, because each must be computed by a C++ program, which is a string, and there are only countably many strings over the alphabet of all symbols used for C++ programs, and we are done.

6. Prove the theorems on the handout CanonEnum.pdf.

   Those proof are in the handout.

7. Review the proof that the halting problem is undecidable.

   That proof is in a handout.

8. What is the Church-Turing thesis?

   Every machine is equivalent to some Turing machine. Alternatively stated, every computation that can be done by any mnachine can be done by a Turing machine.

9. Give a definition of the class $\mathcal{NC}$.

   A language L is defined to be $\mathcal{NC}$ if L is accepted in polylogarithmic time by polynomially many processors working in parallel.

10. Give a definition of the class $\mathcal{P}$–complete.

    A language L_1 is defined to be $\mathcal{P}$–complete if, for any $\mathcal{P}$–TIME language L_2, there is an $\mathcal{NC}$ reduction of L_2 to L_1.

11. Name a language which is known to be $\mathcal{P}$–complete.

    The Boolean Circuit Problem.

12. Give a definition of the class $\mathcal{P}$–SPACE–complete.

    A language L is $\mathcal{P}$–SPACE if it is decided by some program that uses decides L using space which is bounded by a polynomial function of its input. A language L_1 is $\mathcal{P}$–SPACE–complete if, given any $\mathcal{P}$–SPACE language L_2, there is a polynomial time reduction of L_2 to L_1.

13. Name a language which is known to be $\mathcal{P}$–SPACE–complete. There are many. For example, the set of configurations of RUSH HOUR from which it is possible to win.

14. Let L be the Dyck language, but where each left parenthesis is written as a and every right parenthesis as b. (This makes grading easier, since if you write parentheses carelessly, they look alike.)

Here is an unambiguous CFG for L.

1. $S \to a_2 S_3 b_4 S_5$
2. $S \to \lambda$

(a) Fill in the action and goto tables for an LALR parser for the grammar given above. I have started the tables by writing row 0 and row 4.

	a	b	\$	S
0	$s2$		$r2$	1
1			**halt**	
2	$s2$	$r2$		3
3		$s4$		
4	$s2$	$r2$	$r2$	5
5		$r1$	$r1$	

(b) Show the computation of the parser for the input string $aabbab$.

$\$_0$	$aabbab\$$		
$\$_0 a_2$	$abbab\$$		s_2
$\$_0 a_2 a_2$	$bbab\$$		s_2
$\$_0 a_2 a_2 S_3$	$bbab\$$	2	r_2
$\$_0 a_2 a_2 S_3$	$bab\$$	2	s_4
$\$_0 a_2 a_2 S_3 b_4$	$bab\$$	2	s_4
$\$_0 a_2 a_2 S_3 b_4 S_5$	$bab\$$	22	r_2
$\$_0 a_2 S_3$	$bab\$$	221	r_1
$\$_0 a_2 S_3 b_4$	$ab\$$	221	s_4
$\$_0 a_2 S_3 b_4 a_2$	$b\$$	221	s_2
$\$_0 a_2 S_3 b_4 a_2 S_3$	$b\$$	2212	r_2
$\$_0 a_2 S_3 b_4 a_2 S_3 b_4$	\$	2212	s_4
$\$_0 a_2 S_3 b_4 a_2 S_3 b_4 S_5$	\$	22122	$r2$
$\$_0 a_2 S_3 b_4 S_5$	\$	221221	$r1$
$\$_0 S_1$	\$	2212211	$r1$

halt 2212211

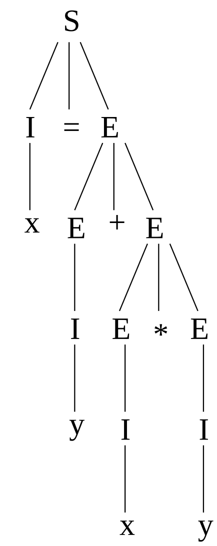
15. The following CF grammar models an assignment statement. We allow just two identifiers, x and y, and two operators $+$ and $*$. We have three grammar variables, S for assignment statement, I for identifier, and E for expression. We have the equal sign as a symbol. The start symbol is S.

1. $S \to I_2 =_3 E_4$
2. $I \to x_5$
3. $I \to y_6$
4. $E \to I_7$
5. $E \to E +_8 E_9$
6. $E \to E *_{10} E_{11}$

	x	y	$+$	$*$	$=$	$\$$	S	I	E
0	$s5$	$s6$					1	2	
1						**halt**			
2					$s3$				
3	$s5$	$s6$						7	4
4			$s8$	$s10$		$r1$			
5			$r2$	$r2$	$r2$	$r2$			
6			$r3$	$r3$	$r3$	$r3$			
7			$r4$	$r4$	$r4$	$r4$			
8	$s5$	$s6$						7	9
9			$r5$	$s10$		$r5$			
10	$s5$	$s6$						7	11
11			$r6$	$r6$		$r6$			

(a) Sketch the parse tree of the string

$x = y + x * y$



(b) Identify the entries of the Action table which ensure that addition and multiplicaion are left associative and that multiplication has precedence over addition.

Row 9 column + ensures that addition is left associative.
Row 11 column $*$ ensures that multiplication is left associative.
Row 9 column $*$ and row 11 column + ensure that multiplication has precedence over addition.

16. (a) Give a CNF grammar for the language L of problem 14.
(b) Use that grammar and the CYK algorithm to prove that $aababb \in L$.

There are several choices of CNF grammar for L. Here is one.

$S \to AB$
$T \to AB$
$A \to a$
$A \to AT$
$B \to b$
$B \to BT$
$S \to \lambda$

$aabbab \in L$ since S is in the top cell of the CYK diagram.