CSC 456/656 Fall 2025 Answers to Third Examination April 9, 2025

Name:____

The entire test is 335 points.

No books, notes, scratch paper, celphones, calculators, or laptops. Use pen or pencil, any color. Use the rest of this page and the backs of the pages for scratch paper. If you need more scratch paper, it will be provided.

1. (i) [5 points] **F** Every subset of a regular language is regular.

   (ii) [5 points] **T** 2-SAT is $\mathcal{P}$–TIME.

   (iii) [5 points] **T** A language L is $\mathcal{NP}$ if and only if there is a polynomial time reduction of L to TSP, the Traveling Salesman Problem.

   (iv) [5 points] '**F** If a computable sequence of fractions converges to x, then x must be a recursive real number.

   (v) [5 points] **T** If L is a recursively enumerable language, there must be a computable reduction of L to the halting problem.

   (vi) [5 points] **F** If two context-free grammars are equivalent, there must be a proof that they are equivalent.

   (vii) [5 points] **T** If two context-free grammars are not equivalent, there must be a proof that they are not equivalent.

   (viii) [5 points] **T** The membership problem for any regular language is $\mathcal{NC}$.

   (ix) [5 points] **T** Given any sequence $M_1, \ldots M_n$ of $n \times n$ Boolean matrices, the product $M_1 \times M_2 \times \cdots \times M_n$ is $\mathcal{NC}$ computable.

   (x) [5 points] **T** The problem of determining whether a given integer is a square is $\mathcal{NC}$.

   (xi) [5 points] **F** The set of all real numbers which are limits of convergent sequences of fractions is countable.

   (xii) [5 points] **T** $\sqrt{2}$ is a recursive real number.

   (xiii) [5 points] **F** Let x be a real number whose decimal digits are all either 0 or 1. then x must be a redursive real number.

   (xiv) [5 points] **T** Every context-sensitive language is decidable.

   (xv) [5 points] **F** $\mathcal{P}$–TIME = EXP–TIME.

   (xvi) [5 points] **F** $\mathcal{NC} = \mathcal{P}$–SPACE.'

   (xvii) [5 points] **O** The time to decide whether a Boolean expression is satisfiable is exponential in the worst case.

   (xix) [5 points] **F** The set of all binary languages is countable.

   (xx) [5 points] **T** $\mathcal{NC}$ contains all context-free languages.

3. [20 points] Prove that every $\mathcal{RE}$ language is accepted by some machine.

Let L be an $\mathcal{RE}$ language. Let $w_1, w_2, \ldots$ be the enumeration of L written by some machine. The following program accepts L:

Read w
For i from 1 to ∞
    If($w = w_i$)
        Accept w

2. [20 points] Prove that the set of recursive real numbers is countable.

There are only countably many strings over any language, henece the set of all C++ programs is countable. Every machine is emulated by some C++ program, hence there are countably many machines. The decimal expansion of each recursive real number is computed by some machine, hence there are only countably many recursive real numbers.

4. [20 points] Prove that the halting problem is undecidable.

Proof: By contradiction. Assume the halting problem is decidable. Let Q be the machine implemented by the following program.

Read a machine desciptrion $\langle M \rangle$.
    If M halts with input $\langle M \rangle$
        Enter an infinite loop.
    Else
        Halt

If Q halts with input $\langle Q \rangle$, then it runs forever.
If Q does not halt with input $\langle Q \rangle$, then it halts.
Either case is a contradiction, hence the halting problem is undecidable. ∎

5. [10 points] What is the Church-Turing thesis?

Every machine is equivalent to some Turing machine.

6. [10 points] Name a language which is known to be $\mathcal{P}$–SPACE–complete.

One example is the set of RUSH HOUR positions from which it is possible to win.

7. [20 points] Give a definition of the class $\mathcal{P}$–SPACE

A language L is $\mathcal{P}$–SPACE if there is a program which decides whether a string w is in L using memory which does not exceed a polynomial function of the length (in bits) of w.

8. [20 points] Fill in the missing column of the Action table for the CFG below:

1. $E \rightarrow E +_2 E_3$
2. $E \rightarrow E -_4 E_5$
3. $E \rightarrow E *_6 E_7$
4. $E \rightarrow -_8 E_9$
5. $E \rightarrow (_{10} E_{11})_{12}$
6. $E \rightarrow x_{13}$

	x	$+$	$-$	$*$	(	)	$	E
0	s13		s8		s10			1
1		s2	s4	s6			**halt**	
2	s13		s8		s10			3
3		r1	r1	s6		r1	r1	
4	s13		s8		s10			5
5		r2	r2	s6		r2	r2	
6	s13		s8		s10			7
7		r3	r3	r3		r3	r3	
8	s13		s8		s10			9
9		r4	r4	r4		r4	r4	
10	s13		s8		s10			11
11		s2	s4	s6		s12		
12		r5	r5	r5		r5	r5	
13		r6	r6	r6		r6	r6	

9. [20 points] Use the pumping lemma for regular languages to prove that $L = \{a^n b^n\}$ is not regular.

Proof: By contradiction. Assume L is regular. By the pumping lemma, we can choose a positive integer p, called a pumping length of L, such that, for each string $w \in L$ of length at least p, there exist strings x, y, z such that:
1. $w = xyz$
2. $|xy| \leq p$
3. $|y| \geq 1$
4. For any integer $i \geq 0$ $xy^iz \in L$.

Let p be a pumping length of L. Let $w = a^pb^p$. Note that $w \in L$, and has length at least p. Let the strings x, y, z be as given by the pumping lemma. By 1. and 2., we know that xyz is a substring of a^n, hence $y = a^j$ for some j. By 3., $j > 0$. Let $i = 0$. By 4., $xz \in L$. xz is obtained by deleting y from w, hence $xz = a^{p-j}b^p$. Since $p - j < p$, $xz \notin L$, contradiction.∎

10. [10 points] Name a language which is known to be $\mathcal{P}$–complete.

There are many examples. One we discussed in class is the Boolean Circuit Problem.

11. [20 points] Let L be the Dyck language, but where each left parenthesis is written as a and every right parenthesis as b. (This makes grading easier, since if you write parentheses carelessly, they look alike.)

Here is an unambiguous CFG for L.

1. $S \rightarrow a_2 S_3 b_4 S_5$
2. $S \rightarrow \lambda$

Fill in the action and goto tables for an LALR parser for the grammar given above. I have started the tables by writing row 0 and row 4.

	a	b	$\$$	S
0	$s2$		$r2$	1
1			**halt**	
2	$s2$	$r2$		3
3		$s4$		
4	$s2$	$r2$	$r2$	5
5		$r1$	$r1$	

12. [20 points] Prove that, if a language L is enumerated by some machine in canonical order, L is decidable.

Proof: Let $w_1, w_2, \ldots$ be the canonical enumeration of L. Since those are written by a machine, we can input them in canonical order. The following program decides L. Read w

For $i = 1$ to ∞

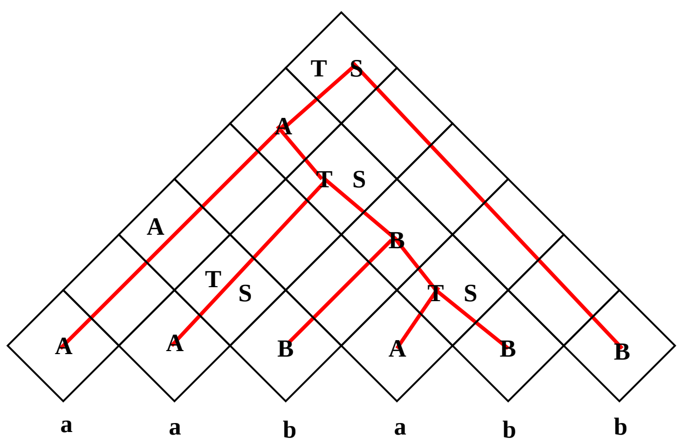
    If $w = w_i$ Halt and Accept w

    If $w < w_i$ in the canonical order

        Halt and Reject w ▮

14. [20 points] The following is a CNF grammar for the language of Problem 11. Use that grammar and the CYK algorithm to prove that $aababb \in L$.

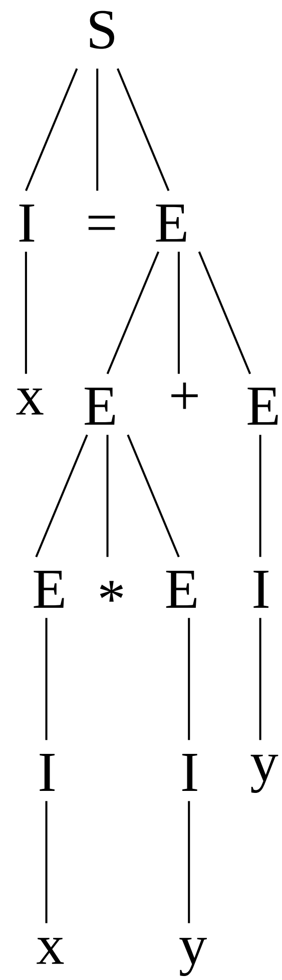
$S \to AB$
$T \to AB$
$A \to a$
$A \to AT$
$B \to b$
$B \to BT$
$S \to \lambda$



13. The following CF grammar models an assignment statement. We allow just two identifiers, x and y, and two operators $+$ and $*$. We have three grammar variables, S for assignment statement, I for identifier, and E for expression. We have the equal sign as a symbol. The start symbol is S.

1. $S \to I_2 =_3 E_4$
2. $I \to x_5$
3. $I \to y_6$
4. $E \to I_7$
5. $E \to E +_8 E_9$
6. $E \to E *_{10} E_{11}$

	x	y	$+$	$*$	$=$	$\$$	S	I	E
0	$s5$	$s6$					1	2	
1						**halt**			
2					$s3$				
3	$s5$	$s6$						7	4
4			$s8$	$s10$		$r1$			
5			$r2$	$r2$	$r2$	$r2$			
6			$r3$	$r3$	$r3$	$r3$			
7			$r4$	$r4$	$r4$	$r4$			
8	$s5$	$s6$						7	9
9			$r5$	$s10$		$r5$			
10	$s5$	$s6$						7	11
11			$r6$	$r6$		$r6$			



(a) [10 points] Sketch the parse tree of the string $y = x * y + y$

(b) [20 points] Identify the entries of the Action table which ensure that addition and multiplicaion are left associative and that multiplication has precedence over addition.

Row 9 column "+" add is left ass. Row 11 column "*" mult is left ass.
Row 9 column "*" and. Row 11 column "+" mult has precedence over add.