

Canonical Ordering and Recursive Enumeration

We define *canonical ordering* on strings as follows.

1. If u, v are strings and $|u| < |v|$, then u is less than v in the canonical ordering.
2. If u, v are string of the same length over an ordered alphabet Σ , we say that u is less than v in the canonical ordering if u comes before v in alphabetic order.

Every language is enumerable, in the sense that an enumeration of the the language exists, but that doesn't mean that an enumeration can be calculated. A language L is defined to be *recursively enumerable* ($\mathcal{RE}$) if there is a machine which enumerates L . There are uncountably many languages over a given alphabet Σ , but only countably many of those are recursively enumerable.

Theorem 1 *If a language L is enumerated in canonical order by some machine, then L is decidable.*

Proof: Let $w_1, w_2, \dots$ be the canonical enumeration of L , which is given by some machine. The following program decides L .

```
Read  $w$ 
For  $i = 1$  to  $\infty$ 
  If  $w_i = w$  HALT ACCEPT
  If  $w_i > w$  in the canonical ordering HALT REJECT
```

■

Theorem 2 *If a language L is recursive (decidable), then L is enumerated in canonical order by some machine.*

Proof: Let Σ be the alphabet of L . Let $w_1, w_2, \dots$ be the canonical enumeration of Σ^* . The following program write an enumeration of L in canonical order.

```
For  $i = 1$  to  $\infty$ 
  If( $w_i \in L$ ) write  $w_i$ 
```

■

Theorem 3 *If a language L is recursively enumerable, then L is accepted by some machine.*

Proof: Suppose L is recursively enumerable. Then there is some program which (possibly running forever) which writes an enumeration of L . Let $w_1, w_2, \dots$ be that enumeration. The following program accepts L :

```
Read  $w$ 
For  $i = 1$  to  $\infty$ 
  If( $w = w_i$ ) HALT and ACCEPT
```

■

Theorem 4 *If a language L is accepted by some machine M , then L is recursively enumerable.*

Proof: Let M be a machine which accepts a language L over an alphabet Σ . The following program enumerates L . Let $w_1, w_2, \dots$ be the canonical ordering of Σ^* . The following program enumerates L .

For $t = 1$ to ∞

For $i = 1$ to t

If (M accepts w_i within t time units)

write w_i

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