Proof Techniques

The majority of UNLV students have never written a proof. We need to change that, at least for this class! I will assume that you understand elementary rules of logic.

Proof by Contradiction

To prove a statement P by contradiction, we first assume P is false, then proceed logically until we reach a conclusion that is false. This will proves that P is true.

According to legend, the man who first proved that the square root of 2 is irrational was Hippasus of Metapontum, a Greek philosopher and follower of Pythagoras. The story claims that the Pythagoreans threw him overboard from a ship as punishment for revealing this discovery, which challenged their beliefs about the nature of numbers.

One problem is that an important proof in Euclid (out of which I studied geometry) assumes that all numbers are rational. Since our book was printed in modern times, it apologized for this assumption, stating that there is a modern proof that does not use that assumption.

Recall that a *rational* number is a number that can be written as a fraction $\frac{p}{q}$ reduced to the lowest terms, that is, p and q are integers whose greatest common divisor is 1.

Theorem 1 *There is no rational number whose square is 2.*

Proof: Assume that there is some fraction $\frac{p}{q}$ equal to the square root of 2.

We assume that the fraction is reduced to the lowest temrs, *i.e.,* p and q have no common divisor greater than 1. Then

$$\begin{aligned}
\sqrt{2} &= \frac{p}{q} \\
2 &= \frac{p^2}{q^2} \\
2q^2 &= p^2 \text{ thus } p^2 \text{ is even} \\
& \quad\ \text{Thus } p \text{ is even} \\
p &= 2k \text{ for some integer } k \\
p^2 &= 4k^2 \\
2q^2 &= 4k^2 \\
q^2 &= 2p^2 \text{ thus } q^2 \text{ is even} \\
& \quad\ \text{Thus } q \text{ is even}
\end{aligned}$$

Since p and q are both even, they have a common divisor of 2, contradiction. Hence our assumption, that $\sqrt{2}$ is rational, is false. ∎

Proof by Induction

The inductive principle states that if a proposition is true for 1 and, if its true for a positive integer n it is true for $n+1$, then it is true for all positive integers.

Theorem 2 *The sum of the first n integers is* $\frac{n(n+1)}{2}$.

Proof: Let $F(n) = 1+2+3+\cdots+n$, the sum of the first n integers, and let $G(n) = \frac{n(n+1)}{2}$. We prove, by induction, that $F(n) = G(n)$ for all positive integers n. The statement is true for $n=1$, since $F(1)1$ and $G(1) = \frac{1(1+1)}{2} = \frac{2}{2}$. The *inductive step* is to prove that $F(n) = G(n)$ implies $F(n+1) = G(n+1)$ for any n. By definition, $F(n+1) = F(n) + n + 1$, and

$$\begin{aligned} G(n+1) &= \frac{(n+1)(n+1+1)}{2} \\ &= \frac{n^2+3n+2}{2} \\ &= \frac{n^2+n}{2} + \frac{2n+2}{2} \\ &= G(n) + (n+1) \\ &= F(n) + (n+1) \\ &= F(n+1) \end{aligned}$$

▮

Theorem 3 *For any positive integer n, the sum of the first n squares is* $\frac{n(n+1)(2n+1)}{6}$

Proof: Let $F(n) = 1^2+2^2+3^2+\cdots+n^2$, the sum of the first n squares. Let $G(n) = \frac{n(n+1)(2n+1)}{6}$ for any n. We prove that $F(n) = G(n)$ for all positive integers. The statement is true for $n=1$, since $F(1) = 1^2 = 1$ and $G(1) = \frac{1\cdot(1+1)\cdot(2+1)}{6} = 1$. The *inductive step* is to prove that $F(n) = G(n)$ implies $F(n+1) = G(n+1)$ for any n. By definition, $F(n+1) = F(n) + (n+1)^2$, and $G(n+1) = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$

$$\begin{aligned} G(n+1) &= \frac{(n+1)(n+1+1)(2(n+1)+1)}{6} \\ &= \frac{2n^3+9n^2+13n+6}{6} \\ &= \frac{2n^3+3n^2+n}{6} + \frac{6n^2+12n+6}{6} \\ &= G(n) + (n+1)^2 \\ &= F(n) + (n+1)^2 \\ &= F(n+1) \end{aligned}$$

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