

More on HW7: The Two Traveling Salesmen Problem

Error

In the homework page, I added the costs for the paths incorrectly. The total cost of the red and blue paths is 26, not 30. I will now take a sabbatical to repeat the second grade. **This was a joke.**

Hints:

If the weights are presented in matrix form, the matrix has n^2 entries, and $\binom{n}{2} = \frac{n(n-1)}{2}$ defined entries. Just to read the matrix takes $\Theta(n^2)$ time. My algorithm, in this case, has exactly $\frac{n(n+1)}{2}$ subproblems, and they are worked in topological order. Examples 1 and 2 show the matrix input, and also the output.

Each subprogram is to compute “The smallest weight of any ... such that ...,” along with whatever appropriate data which enable you to reconstruct the two paths after all subproblems are solved. I worked the subproblems in topological order. The directed graph of subproblems is *not* complete, which implies that there are multiple topological orders.

My algorithm runs forward, namely 1 to n , but it can be modified to run backward, n to 1. I have not yet written the backward version. The backward version has different subproblems, although the number of subproblems is the same. In fact, the directed graph of subproblems for the backward version looks exactly like the the one for the forward version, with the arrows reversed.

If G is sparse, meaning that m is substantially less than $\binom{n}{2}$, as in Example 2, you can save time and space. (We assume that $m \geq n - 1$, since otherwise, there is no path from 1 to n .) I can modify my algorithm so that it has $n + m$ subproblems, and so that the time complexity is $O(m)$. However, I simply used the $\Theta(n^2)$ -time algorithm for Example 2, since I have not yet implemented the faster version.

A sparse weighted directed graph can be represented as an array of either in-neighbor lists or out-neighbor lists. There is an $O(m)$ -time algorithm to convert one to the other. And, of course, you can convert either of those representations to the matrix representation, or vice-versa, in $O(n^2)$ time.

I have not yet implemented either the forward or the backward fast, *i.e.*, $O(m)$ -time, version. I have given some thought to what kind of data structures would be needed. The set of subproblems is the same, but most of them are useless, and thus the data structure of subproblems you would compute should also be sparse, since otherwise the program would take $\Theta(n^2)$ time despite the sparse input.

Examples:

Let G have n vertices, named $1, 2, \dots, n$, in topological order. Thus, there is no edge (i, j) if $i \geq j$.

Example 1: Let $n = 20$, and let G be a complete acyclic directed graph, *i.e.*, there is an edge from i to j if and only if $i < j$. Let the weights $\{W(i, j)\}$ be given by the following table; for example, $W(1, 2) = 5$. A blank entry means that $W(i, j)$ is undefined, or ∞ if you prefer.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1		5	6	7	8	6	8	5	8	4	6	6	4	4	3	9	1	9	3	4
2			5	8	8	5	5	6	4	9	8	1	4	1	6	1	4	8	9	8
3				2	8	8	9	5	8	8	5	5	8	7	2	7	6	6	2	5
4					9	7	5	1	4	5	3	2	7	7	5	2	6	2	1	3
5						6	2	6	3	6	4	9	9	7	5	4	6	4	9	7
6							1	1	9	7	6	7	2	2	4	2	7	3	6	4
7								3	6	1	4	6	9	9	1	6	2	6	2	9
8									6	6	3	4	5	5	3	6	2	4	1	7
9										6	3	2	8	2	9	4	6	8	1	1
10											6	8	7	1	9	9	8	5	5	9
11												1	6	5	7	1	1	4	3	9
12													2	4	3	5	2	7	3	8
13														4	9	1	2	7	6	6
14															8	3	5	5	9	1
15																9	6	1	9	3
16																	3	1	7	8
17																		2	6	7
18																			8	3
19																				3
20																				

My program computed the following two paths.

Start at 1

Move to 2 at cost 5

Move to 3 at cost 5

Move to 4 at cost 2

Move to 8 at cost 1

Move to 9 at cost 6

Move to 11 at cost 3

Move to 12 at cost 1

Move to 13 at cost 2

Move to 16 at cost 1

Move to 17 at cost 3

Move to 19 at cost 6

Move to 20 at cost 3

Total cost: 67

Start at 1

Move to 5 at cost 8

Move to 6 at cost 6

Move to 7 at cost 1

Move to 10 at cost 1

Move to 14 at cost 1

Move to 15 at cost 8

Move to 18 at cost 1

Move to 20 at cost 3

Example 2: Let $n = 20$, and let G be a sparse acyclic directed graph. Let the weights $\{W(i, j)\}$ be given by the following table. There are many cases of $i < j$ for which $W(i, j)$ is undefined, for example, $W(8, 10)$ is undefined.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1		6																		
2			8		6	9														
3				8		5														
4					2			2												
5						5	6	9												
6							6	3		4										
7								3	5	3										
8									4			1								
9										9	6	7	1							
10											7	8	9	2						
11												9	9		9					
12													6			3				
13														2		1	7			
14															3		3			
15																9			9	
16																	5	8		
17																		3		
18																			4	
19																				3
20																				

My program computed the following two paths.

Start at 1

Move to 2 at cost 6
Move to 5 at cost 6
Move to 6 at cost 5
Move to 7 at cost 6
Move to 10 at cost 3
Move to 11 at cost 7
Move to 12 at cost 9
Move to 16 at cost 3
Move to 17 at cost 5
Move to 18 at cost 3
Move to 19 at cost 4
Move to 20 at cost 3
Total cost: 106

Start at 1

Move to 2 at cost 6
Move to 3 at cost 8
Move to 4 at cost 8
Move to 8 at cost 2
Move to 9 at cost 4
Move to 13 at cost 1
Move to 14 at cost 2
Move to 15 at cost 3
Move to 19 at cost 9
Move to 20 at cost 3

Example 3: Let $n = 20$, and let G be another sparse acyclic directed graph, given by the following table. Unlike the previous examples, there is not always a path from i to $i + 1$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1		7																		
2			4																	
3				5	6	1	6													
4					4	7	6	3												
5							1		2											
6							7	1	3											
7								5		9										
8									6		8	1								
9										6										
10											7									
11												5	5	7						
12													3	6	6					
13														9	4	4				
14															1					
15																2		2		
16																	6	4	1	
17																		5	7	8
18																			9	8
19																				1
20																				

Start at 1
Move to 2 at cost 7
Move to 3 at cost 4
Move to 4 at cost 5
Move to 5 at cost 4
Move to 7 at cost 1
Move to 8 at cost 5
Move to 12 at cost 1
Move to 13 at cost 3
Move to 16 at cost 4
Move to 19 at cost 1
Move to 20 at cost 1

Total cost: 93

Start at 1
Move to 2 at cost 7
Move to 3 at cost 4
Move to 6 at cost 1
Move to 9 at cost 3
Move to 10 at cost 6
Move to 11 at cost 7
Move to 14 at cost 7
Move to 15 at cost 1
Move to 16 at cost 2
Move to 17 at cost 6
Move to 18 at cost 5
Move to 20 at cost 8

Example 4. $n = 25$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1		11	12	15	17	13																			
2			18	11	17	9	13																		
3				13	8	9	6	18																	
4					2	20	6	8	10																
5							11		8																
6							17																		
7								1		20	3														
8										20		17													
9										11	14		13												
10											3	19	11												
11												10			15										
12													5	3											
13														3	5	8	19								
14															9		9	14							
15																2	14	14							
16																	8		7	8					
17																		12			20	20			
18																			14			4	20	14	
19																				6		10	10	5	
20																					4	7	3	14	14
21																						7	3	16	4
22																							19	8	14
23																								17	1
24																									2
25																									

Start at 1
Move to 2 at cost 11
Move to 4 at cost 11
Move to 5 at cost 2
Move to 9 at cost 8
Move to 10 at cost 11
Move to 11 at cost 3
Move to 15 at cost 15
Move to 16 at cost 2
Move to 19 at cost 7
Move to 20 at cost 6
Move to 23 at cost 3
Move to 25 at cost 1
Total cost: 186

Start at 1
Move to 3 at cost 12
Move to 6 at cost 9
Move to 7 at cost 17
Move to 8 at cost 1
Move to 12 at cost 17
Move to 13 at cost 5
Move to 14 at cost 3
Move to 17 at cost 9
Move to 18 at cost 12
Move to 21 at cost 4
Move to 22 at cost 7
Move to 24 at cost 8
Move to 25 at cost 2

Example 5. $n = 25$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1		11	12	15	17	13																			
2			18	11	17	9	13																		
3				13	8	9	6	18																	
4					2	20	6	8	10																
5						17	10	8	17																
6							7		1																
7								20	3	18	10														
8									17		16	3													
9										13	3	19	11												
10											9														
11												15			3										
12													7	3	5	8									
13														19	9		9								
14															14			14							
15																14									
16																	16		8						
17																				20	20				
18																			14		4	20	14		
19																						9	10	10	
20																					13	4	7	3	14
21																						14	7	3	16
22																							4	19	8
23																								14	17
24																									1
25																									

There is no solution.

Example 6. $n = 25$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1		4	14	7	6	14																			
2			5	2	19	13	7																		
3				10	9	6	13	5																	
4					18	11	9	8	19																
5						12			12																
6										7															
7								12	14																
8									20	8		19													
9										14	6	12	20												
10											3	6													
11													2												
12																18									
13														4	9										
14															12	19	19	2							
15																12	18		8						
16																			14						
17																		18	1			10			
18																			12			2	4		
19																				18	2	6	10	10	
20																					10	10	11	20	20
21																						9	2	18	1
22																							3	10	6
23																								10	9
24																									5
25																									

Start at 1

Move to 2 at cost 4

Move to 4 at cost 2

Move to 7 at cost 9

Move to 8 at cost 12

Move to 9 at cost 20

Move to 11 at cost 6

Move to 13 at cost 2

Move to 14 at cost 4

Move to 15 at cost 12

Move to 17 at cost 18

Move to 18 at cost 18

Move to 22 at cost 2

Move to 23 at cost 3

Move to 24 at cost 10

Move to 25 at cost 5

Total cost: 231

Start at 1

Move to 2 at cost 4

Move to 3 at cost 5

Move to 5 at cost 9

Move to 6 at cost 12

Move to 10 at cost 7

Move to 12 at cost 6

Move to 16 at cost 18

Move to 19 at cost 14

Move to 20 at cost 18

Move to 21 at cost 10

Move to 25 at cost 1

Example 7: $n = 25$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1			11		9	19																			
2			18																						
3				20		18	10																		
4					7	18	19	20																	
5						20		14	5		5														
6							4	19	6																
7								5	11	9		9													
8									2	15		4	7												
9										1				12											
10											6	19		20	19										
11												19		6											
12													1	19		8	17	15							
13														20			18	16							
14																13				5					
15																2				9					
16																	15	14							
17																		2					2		
18																			13	15	4			15	
19																				14	18	12	1		
20																					2		6	15	
21																								3	17
22																						10		18	15
23																								9	
24																									
25																									

There is no solution.