Graphs and Directed Graphs

Graphs and directed graphs have so many practical and theoretical applications that (I'm almost certain) no one has listed them all.

Graphs

Definition 1 A graph is an ordered pair (V, E) where V is a finite set and E is a set of unordered pairs of members of V.

For a graph G = (V, E), the members of V are called *vertices* of G and the members of E are called *edges* of G. We illustrate two graphs below.



We can write a graph in purely set notation, for example:

 $G_1 = \left(\{A, B, C, D, E, F, G, H\}, \{\{B, F\}, \{A, H\}, \{B, H\}, \{B, D\}, \{C, H\}, \{C, D\}, \{C, E\}, \{E, G\}\}\right)$ Exercise 1 Write G_2 in purely set notation.

Definition 2 A path of a graph G = (V, E) is a finite sequence of vertices $\sigma = v_0, v_1, \ldots v_n$ where any two consecutive vertices form and edge, i.e., $\{v_{i-1}, v_i\} \in E$

In Definition 2 above, the path σ is from v_0 to v_n , and the length of σ is n.

Definition 3 A path is a cycle if it begins and ends at the same vertex.

Definition 4 A graph is acyclic if it has no cycle.

Definition 5 A graph G is connected if, given any $u, v \in V$, there is a path in G from u to v.

Additional Graph-Related Terms

- 1. Subgraph.
- 2. Component.
- 3. Simple path.
- 4. Simple cycle.
- 5. Diameter.
- 6. Clique.
- 7. Planar graph.
- 8. Weighted graph.
- 9. Bipartite graph.

Directed Graphs

Definition 6 A directed graph is an ordered pair (V, A) where V is a finite set and A is a set of ordered pairs of members of V. The members of A are called directed edges, or arcs. We allow a **trivial** arc (v, v) for any $v \in V$.

Definition 7 A path (or directed path) in a directed graph (V, A) is a sequence of vertices $v_0, v_1, \ldots v_n$, where each (v_{i-1}, v_i) is an arc.

Definition 8 A directed graph G = (V, A) is strongly connected there is a path from any vertex to any other vertex.

Definition 9 A cycle in a directed graph G = (V, A) is a path which begins and ends at the same vertex. G is called acyclic if it has no cycle.

Definition 10 A topological order of a directed graph G = (V, A) is an order of V such that, for any non-trivial arc (u, v) of G, u is before v in that order.

Exercise 2 Fill in the blank: "A directed graph G has a topological order if and only if G is "______."

Additional Directed Graph-Related Terms

- 1. Digraph.
- 2. Subgraph.
- 3. Strong component.
- 4. Simple path.
- 5. Simple cycle.
- 6. Transitive closure.
- 7. Transitive reduction.
- 8. Weighted digraph.
- 9. Dag.
- 10. Hamiltonian cycle.