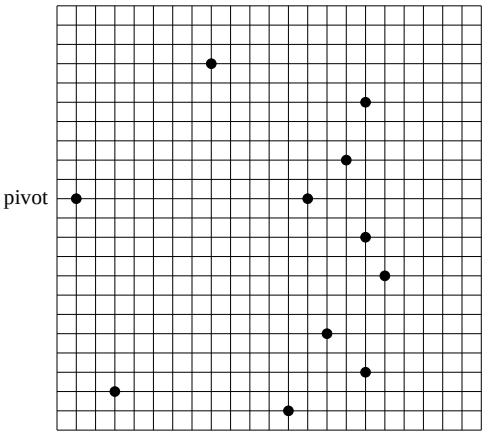
Convex Hulls and Graham Scan

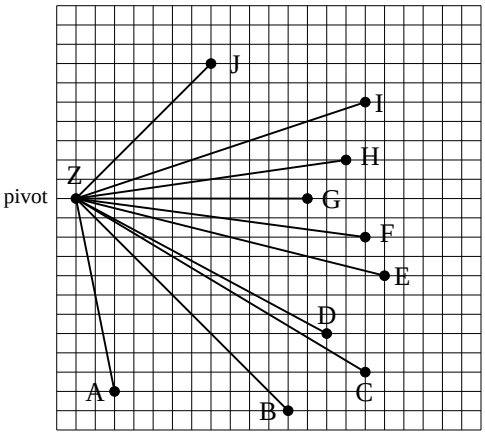
A set $K \subset \mathbb{R}^n$ is *convex* if, given any points $P, Q \in K$, the line segment connecting P and Q is a subset of K. The convex hull of a set of a any set of points in $\mathbb{R}^n$ is the smallest convex set which contains the points. If $K \subset \mathbb{R}^2$, the plane the output of the algorithm Graham scan is the sequence of vertices of the boundary of the convex hull, which is a convex polygon. Graham Scan generates a sequence of vertices. Each vertex is pushed onto a stack, but before any vertex is pushed, one or more vertics may be popped.

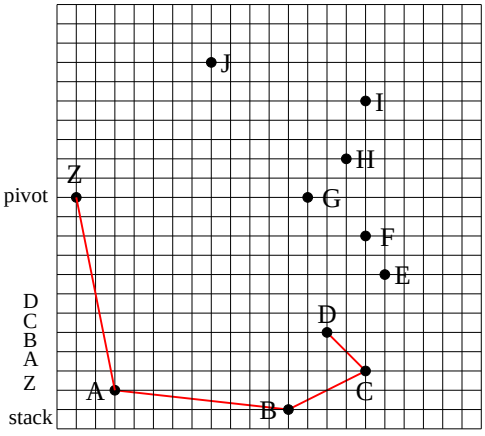
$\overrightarrow{AB} \times \overrightarrow{BC}$

We walk through Graham scan for the set of points shown below. The first step is to pick the pivot, which must be an extreme point of the set, in this case the leftmost point of K.



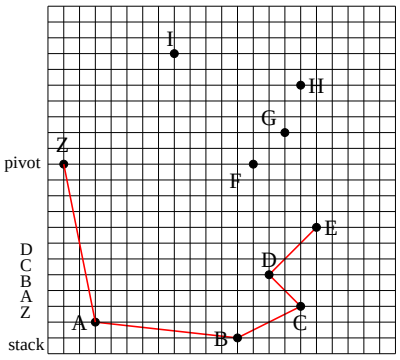
Draw a line from the pivot to each other point of K. Sort the points by the slope of that line. We Label the pivot Z, and the other points A through I.



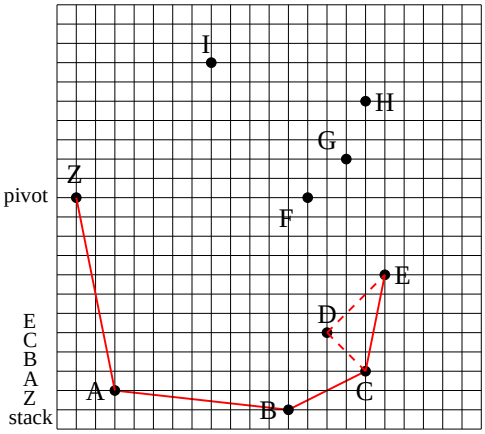


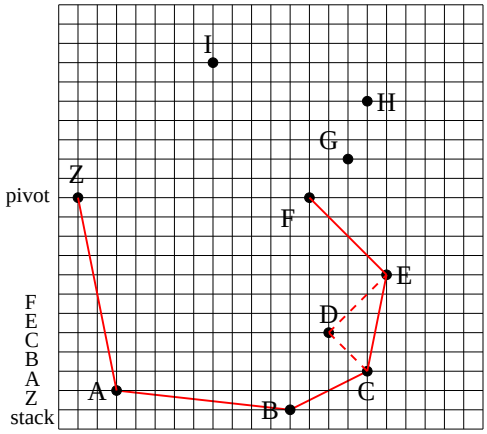
We have built 4 edges of a path starting at Z. each added edge causes the path to make a "left turn," detected

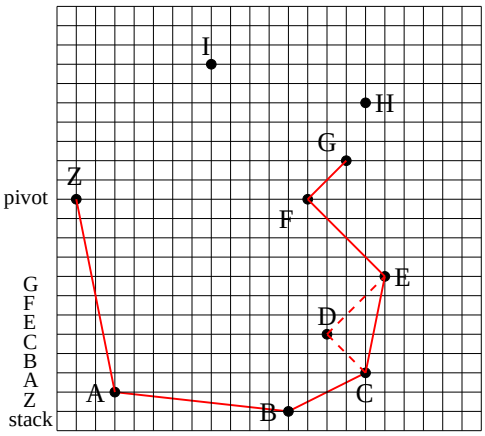
by the fact that the cross product from each vector to the next is negative. For example: $\overrightarrow{ZA} \times \overrightarrow{AB} = \begin{vmatrix} 2 & -10 \\ 10 & -1 \end{vmatrix} = 98$ which is positive, indicating the path is turning left at A. However, $\overrightarrow{CD} \times \overrightarrow{DE} = \begin{vmatrix} -2 & 2 \\ 3 & 3 \end{vmatrix} = -12$ which is negative, indicating the path is turning right at D. Thus, D must be popped before E can be pushed, as shown in the next figure.



Cross Product Decides Turn In Graham Scan, we need to determine whether two 2-dimensional vectors are parallel, or that one turns right or left relative to the other. In our example, the vector



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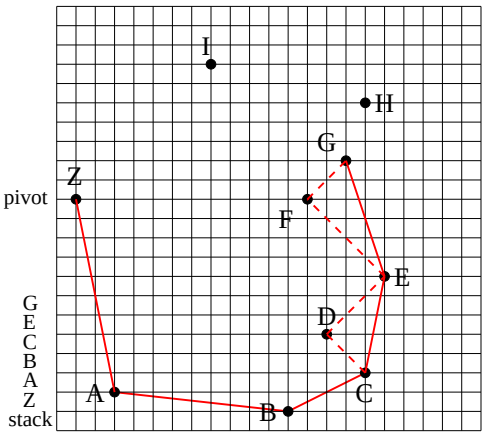
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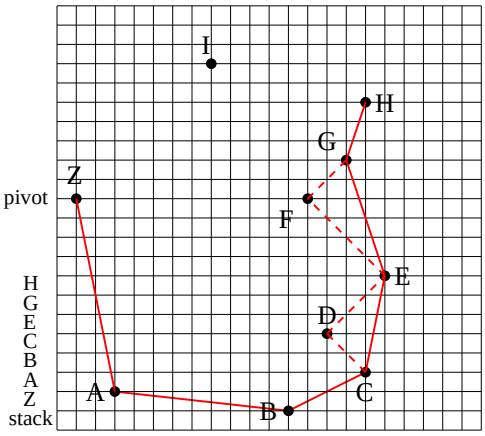
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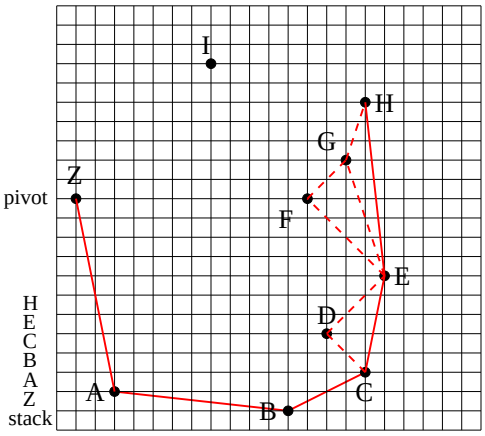
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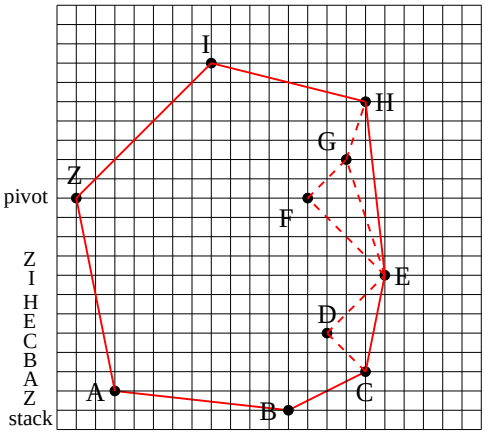
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