# The Fast Fourier Transform

## 1 Complex Numbers

A *complex number*, or *Gaussian number*, is an ordered pair of real numbers $(a, b)$, together with the following operations.

1. (Addition) $(a, b) + (c, d) = (a + c, b + d)$.
2. (Multiplication) $(a, b)(c, d) = (ac - bd, ad + bc)$.

Write $\mathbb{C}$ for the system of all complex numbers, together with the two operations of addition and multiplication.

**Lemma 1.1**

(a) $\mathbb{C}$ *is a field. More specifically:*

1. *The additive identity is* $(0, 0)$.
2. *The additive inverse of* $(a, b)$ *is* $(-a, -b)$.
3. *The multiplicative identity is* $(1, 0)$.
4. *If* $(a, b) \neq (0, 0)$*, the multiplicative inverse of* $(a, b)$ *is* $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2,b^2}\right)$.
5. $\mathbb{C}$ *satisfies the field axioms.*

(b) *The field of real numbers,* $\mathbb{R}$*, embeds in* $\mathbb{C}$ *by* $x \mapsto (x, 0)$.

By a slight abuse of notation, we will consider $\mathbb{R}$ to be a subfield of $\mathbb{C}$, by identifying the real number $x$ with the complex number $(x, 0)$. We assign the special name $\mathbf{i} = (0, 1)$. Thus, we can write the complex number $(a, b)$ as $a + b\mathbf{i}$; we call this *standard form*.

If $z = a + b\mathbf{i}$ is any complex number, we write

1. $|z| = \sqrt{a^2 + b^2}$, the *absolute value* of $z$.
2. $\bar{z} = a - b\mathbf{i}$, the *conjugate* of $z$.
3. $arg(z) = \theta$, where $\theta$ is the angle such that

$$\begin{aligned} a &= |z| \cos\theta \\ b &= |z| \cos\theta \end{aligned}$$

the *argument* of $z$.

We can also write $arg(z) = \begin{cases} \arctan(b/a) \text{ if } a > 0 \\ \arctan(b/a) + \pi \text{ if } a < 0 \text{ and } b \geq 0 \\ \arctan(b/a) - \pi \text{ if } a < 0 \text{ and } b < 0 \\ \frac{\pi}{2} \text{ if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2} \text{ if } a = 0 \text{ and } b < 0 \\ \text{undefined if } a = 0 \text{ and } b = 0 \end{cases}$

We list properties of $\mathbb{C}$. For any $z, w \in \mathbb{C}$:

1. $z\overline{z} = |z|^2$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $\overline{zw} = \overline{z}\,\overline{w}$
4. $|z + w| \leq |z| + |w|$
5. If $z \neq 0$ and $w \neq 0$, then $arg(zw) = arg(z) + arg(w)$.

## 1.1 Roots of Unity

We say that a complex number $z$ is an $n^{\text{th}}$ root of unity if $z^n = 1$. For any $n \geq 1$, there are exactly $n$ $n^{\text{th}}$ roots of unity. If $n \geq 2$, define $\omega_n = \cos\left(\frac{2\pi}{n}\right) + \mathbf{i}\sin\left(\frac{2\pi}{n}\right)$. the $n$ $n^{\text{th}}$ roots of unity are $\{\omega_n^i : 0 \leq i < n\}$. Note that $arg(omega_n) = \frac{2\pi}{n}$.

Questions:

1. What are the $n^{\text{th}}$ roots of unity for $n = 2$?
2. What are the $n^{\text{th}}$ roots of unity for $n = 3$?
3. What are the $n^{\text{th}}$ roots of unity for $n = 4$?
4. The complex number $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{i}$ is an $n^{\text{th}}$ root of unity for which $n$? (The smallest $n$, please.)
5. Find the two square roots of $3 + 4\mathbf{i}$.
6. Rewrite $\frac{3}{2+\mathbf{i}}$ in standard form.