The Lower Bound Proof for the Randomized Competitiveness of the Underemployed Watchman Problem

We will assume that the adversary is oblivious. That means that she does not know where the watchman is, but she sees the watchman's program, and thus can compute p_A, the probability that the watchman is at A.

Theorem 1 *If $C < \frac{3}{2}$, there is no C-competitive randomized online algorithm for the watchman.*

Proof: Let *alg* be any randomized online algorithm for the watchman. For any $\varepsilon > 0$, we define an adversary adv_ε, as follows. (Write r for the request.)

- If the offset function is $(0, 1)$, then

$$r = \begin{cases} A \text{ if } p_A \leq 1 - \varepsilon \\ B \text{ otherwise} \end{cases}$$

- If the offset function is $(0, 0)$, then

$$r = \begin{cases} A \text{ if } p_A < \frac{1}{2} \\ B \text{ otherwise} \end{cases}$$

- If the offset function is $(1, 0)$, then

$$r = \begin{cases} A \text{ if } p_A < \varepsilon \\ B \text{ otherwise} \end{cases}$$

We now define

$$\varepsilon = \frac{\frac{3}{2} - C}{3}$$

which is positive since $C < \frac{3}{2}$.

Lemma 1 *If ϱ is a request sequence of length N generated by adv_ε, then*

$$cost_{alg}(\varrho) \geq C \cdot cost_{\mathrm{OPT}}(\varrho) + \varepsilon N - \tfrac{1}{2}$$

Proof of Lemma: We define three types of requests, which we call *active*, *post-active*, and *lazy*.

- If the offset function is $(0, 1)$, then a is a *lazy* request and b is an *active* request.
- If the offset function is $(1, 0)$, then b is a *lazy* request and a is an *active* request.
- If the offset function is $(0, 0)$, then any request is a *post-active* request.

Every active request (unless it is the last request) is followed immediately by a post-active request. Measuring optimal costs using the amortized costs of the opt-graph, we have:

- The optimal cost for a lazy or post-active request is zero.

- The optimal cost for an active request is 1.
- The algorithm cost for a lazy request is at least ε.
- The algorithm cost for an active request is at least $1 - \varepsilon$.
- The algorithm cost for a post-active request is at least $\frac{1}{2}$.

Let I be the number of active requests, J the number of post-active requests, and L the number of lazy requests, of ϱ. Then

$$\begin{aligned}
J &= I - 1 \quad \text{if the last request is active} \\
J &= I \quad \text{if the last request is not active} \\
N &= L + I + J \\
&\geq L + 2I \\
cost_{\text{OPT}}(\varrho) &= I \\
cost_{alg}(\varrho) &\geq \varepsilon L + (1-\varepsilon)I + \tfrac{1}{2}J \\
&\geq \varepsilon L + (\tfrac{3}{2} - \varepsilon)I - \tfrac{1}{2}
\end{aligned}$$

Thus

$$\begin{aligned}
cost_{alg}(\varrho) - C \cdot cost_{\text{OPT}}(\varrho) &= cost_{alg}(\varrho) - (\tfrac{3}{2} - 3\varepsilon)cost_{\text{OPT}}(\varrho) \\
&\geq \varepsilon L + (\tfrac{3}{2} - \varepsilon)I - \tfrac{1}{2} - (\tfrac{3}{2} - 3\varepsilon)I \\
&= \varepsilon(L + 2I) - \tfrac{1}{2} \\
&\geq \varepsilon N - \tfrac{1}{2}
\end{aligned}$$

This completes the proof of Lemma 1

Returning to the proof of the theorem, suppose that *alg* is C-competitive. Then, there exists a K such that, for any request sequence ϱ,

$$cost_{alg}(\varrho) \leq C \cdot cost_{\text{OPT}}(\varrho) + K$$

Let ρ be the request sequence generated by adv_ε of length N, where

$$N > \frac{K + \frac{1}{2}}{\varepsilon}$$

By Lemma 1

$$\begin{aligned}
cost_{alg}(\varrho) &\geq C \cdot cost_{\text{OPT}}(\varrho) + \varepsilon N - \tfrac{1}{2} \\
&> C \cdot cost_{\text{OPT}}(\varrho) + \varepsilon \frac{K + \frac{1}{2}}{\varepsilon} - \tfrac{1}{2} \\
&= C \cdot cost_{\text{OPT}}(\varrho) + K
\end{aligned}$$

which is a contradiction. □