The Eager Watchman is 2-Competitive

Theorem 1 *The Eager Watchman is 2-Competitive.*

Proof: In order to prove C-competitiveness of an algorithm $\mathcal{A}$, we can first define a potential Φ which is always non-negative between steps, where the following inequality, which we call the *update condition*, holds for each step:

$$\Phi' + \Delta cost_{\mathcal{A}} \leq \Phi + C\Delta cost_{adv}$$

where Φ is the potential before the step and Φ' is the potential after the step. Then, we let $\Delta cost^t_{\mathcal{A}}$ be the watchman's cost at step t, let $\Delta cost^t{}_{adv}$ be the adversary's cost at step t, and let Φ^t be the potential after t steps. We have

$$\begin{aligned} cost_{\mathcal{A}} &= \sum_{t=1}^{n} \Delta cost^t_{\mathcal{A}} \\ cost_{adv} &= \sum_{t=1}^{n} \Delta cost^t_{adv} \\ \Delta cost^t_{\mathcal{A}} + \Phi^t &\leq \Phi^{t-1} + C\Delta cost^t_{adv} \text{ for all } t \end{aligned}$$

Taking the summation of the update condition over all t, we obtain

$$\begin{aligned} cost_{\mathcal{A}} + \sum_{t=1}^{n-1} \Phi^t + \Phi^n &\leq \Phi^0 + \sum_{t=1}^{n-1} \Phi^t + C\,cost_{adv} \\ cost_{\mathcal{A}} &\leq C\,cost_{adv} + \Phi^0 - \Phi^n \\ &\leq C\,cost_{adv} + \Phi^0 \end{aligned}$$

since $\Phi^n \geq 0$. By definition of C-competitiveness, we are done, since Φ^0 is a constant.

Returning to the *Underemployed Watchman* problem, each step consists of two substeps:

1. One light flashes and the adversary services that request.
   (a) If the adversary is on the same side as the light, she pays nothing.
   (b) If the adversary is not on the same side as the light, she can either stay in the same state or change states. In either case, she pays 1.
2. The watchman services the request.
   (a) If the watchman is on the same side as the light, he pays nothing.
   (b) If the watchman is not on the same side as the light, he can either stay in the same state or change states. In either case, he pays 1.

We define a potential Φ between steps as follows.

$$\Phi = \begin{cases} 0 \text{ if the watchman and the adversary are in the same state} \\ 1 \text{ if the watchman and the adversary are in different states} \end{cases}$$

The best way to think of the potential is as the watchman's "blind trust." It's his money, and he uses it to make payments. When the adversary pays, she deposits money into the trust, but at the exchange rate of C:1. That is, if the adversary pays 1, C is deposited into the trust. The watchman is not allowed to query the trust, *i.e.*, he is not told how much money is there, yet he must make sure that the trust balance is never negative.

We now need to verify the update condition for every possible step. Recall that $C = 2$.

1. **The watchman and the adversary are in the same state and the nearer light flashes.** Neither the watchman nor the adversary pay, and the potential remains 0. (See Figure 1.)

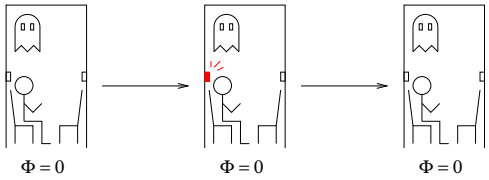


Figure 1: Case 1

2. **The watchman and the adversary are in the same state, the farther light flashes, and the adversary moves.** The watchman moves. The watchman pays 1, the adversary pays 1, and the potential remains 0. The update inequality is a strict inequality. (See Figure 2.)

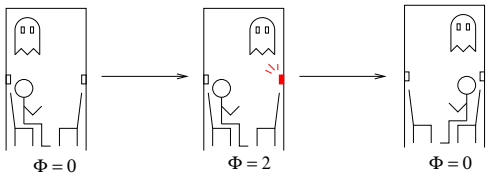


Figure 2: Case 2

3. **The watchman and the adversary are in the same state, the farther light flashes, and the adversary does not move .** The watchman moves. The watchman pays 1, the adversary pays 1, and the potential changes from 0 to 1. The update inequality is an equality. (See Figure 3.)

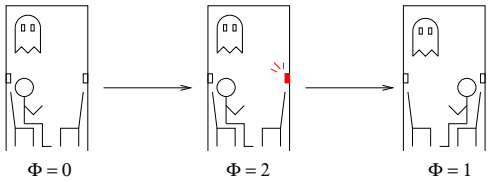


Figure 3: Case 3

4. **The watchman and the adversary are in different states, and the light opposite the watchman flashes.** The watchman moves, and the adversary stays. The watchman pays 1, the adversary pays 0, and the potential changes from 1 to 0. The update inequality is an equality. (See Figure 4.)

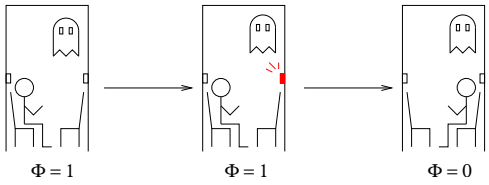


Figure 4: Case 4

5. **The watchman and the adversary are in different states, the light nearer the watchman flashes, and the adversary moves.** The watchman pays 0, the adversary pays 1, and the potential changes from 1 to 0. The update inequality is a strict inequality. (See Figure 5.)

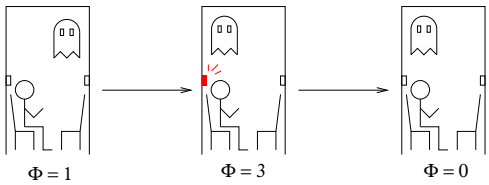


Figure 5: Case 5

6. **The watchman and the adversary are in different states, the light nearer the watchman flashes, and the adversary stays.** The watchman pays 0, the adversary pays 1, and the potential remains 1. The update inequality is a strict inequality. (See Figure 6.)

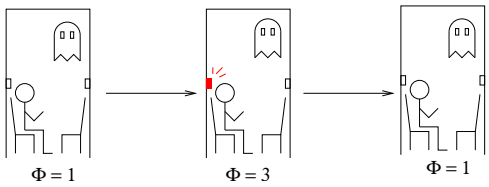


Figure 6: Case 6

In each figure, we show the value of Φ after the first substep, defined to be the potential before the request, plus 2 times the adversary cost, before the watchman's cost is debited.

Note that in Cases 2, 5, and 6, since the update inequality is strict, money is "wasted," that is, in the second substep, the potential is decreased by more than the watchman's cost. That means that those cases are not optimal for the adversary, since the watchman has "money to burn." □