3 Potentials

Potentials are a commonly used tool for proving performance, for example, for proving competitiveness.

3.1 Alternating Weighted Directed Graphs

A common problem in the theory of weighted directed graphs is detection of negative cycle. We generalize this to alternating graphs.

Consider the weighted directed graph, G, shown below (ignore the green boxes for now). The *weight* of each edges is shown as a number on he edge.

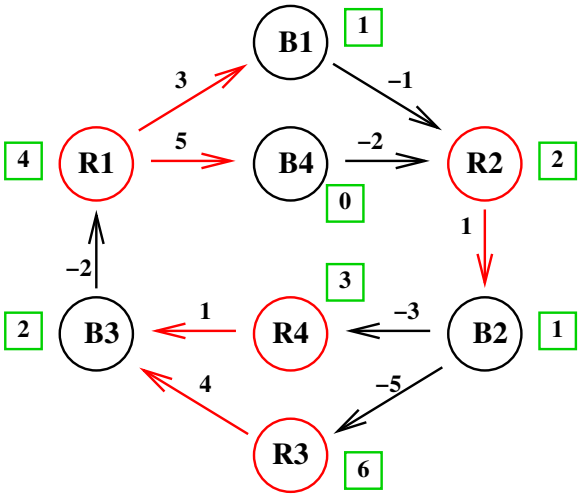


Figure 3.1: Alternating Weighted Graph with Potentials

The edges and nodes of G are colored, either red or black. Note that cycles of G can have positive, negative, or zero weight. Think of G as representing a game between two players, whom we call Red and Black. The nodes of G are the states of the game. (We can think of a pebble resting on one of the nodes.) If the state of the game is red, then Red chooses a new state by following an arrow from the current state; similarly for Black. (Think of the two players taking turns pushing the pebble.) Red's goal is minimize the total weight of the path. Since the game can last arbitrarily long, we say that Red "wins" if the total weight of the path is bounded. This means, that Red must avoid *positive* cycles.

Theorem 3.1 *For any alternating weighted graph G, Red wins if and only if there exists a potential function ϕ on the nodes of G, such that the following three conditions hold:*

1. *$\phi(x) \geq 0$ for each node $x \in G$.*
2. *If x is a red node, then there is some edge (x, y) of G such that $\phi(x) \geq \phi(y) + w(x, y)$.*
3. *If x is a black node, then for any edge (x, y) of G, $\phi(x) \geq \phi(y) + w(x, y)$.*

Exercise 3.1 *Prove Theorem 3.1.*

Exercise 3.2 *Using Theorem 3.1, prove that Red wins the game shown in the figure.*