The Lower Bound Proof for the Randomized Competitiveness of the Underemployed Watchman Problem

We will assume that the adversary is oblivious. That means that she does not know where the watchman is, but she sees the watchman's program, and thus can compute p_A, the probability that the watchman is at A.

Theorem 1 *If $C < \frac{3}{2}$, there is no C-competitive randomized algorithm for the watchman.*

Proof: For any $\varepsilon > 0$, we define an adversary adv_ε, as follows. Write r for the request.

- If the offset function is $(0, 1)$, then

$$r = \begin{cases} A \text{ if } p_A \leq 1 - \varepsilon \\ B \text{ otherwise} \end{cases}$$

- If the offset function is $(0, 0)$, then

$$r = \begin{cases} A \text{ if } p_A < \frac{1}{2} \\ B \text{ otherwise} \end{cases}$$

- If the offset function is $(1, 0)$, then

$$r = \begin{cases} A \text{ if } p_A < \varepsilon \\ B \text{ otherwise} \end{cases}$$

Lemma 1 *No randomized watchman is less than $\left(\frac{3}{2} - \varepsilon\right)$-competitive against adv_ε.*

Proof of Lemma: We define a *phase* to be a sequence of steps beginning and ending with offset function $(0, 0)$. The adversary's cost for each phase is 1.

We show that the watchman pays at least $\frac{3}{2} - \varepsilon$ for each phase, unless the phase is never completed. For the first request of a phase, the watchman pays at least $\frac{1}{2}$. After that request, the offset function is either $(0, 1)$ or $(1, 0)$. Without loss of generality, it is $(1, 0)$.

The adversary then makes requests at B until $p_A < \varepsilon$, and then requests B, completing the phase. The algorithm pays at least $1 - \varepsilon$ for that request, and we are done.

This ends the proof of the Lemma.

We now return to the proof of the Theorem.

Suppose that the watchman is C-competitive against adv_ε, for some $C < \frac{3}{2} - \varepsilon$. Then there exists a constant K such that

$E(cost_{alg}) \leq C \cdot E(cost_{\text{OPT}}) + K$

We will get a contradiction.

The adversary first makes a request at B, paying 1, and bringing the offset function to $(0, 0)$.

After N subsequent phases, the adversary has paid $N + 1$, while the algorithm has paid at least $1 + N\left(\frac{3}{2} - \varepsilon\right)$. For sufficiently large N, we obtain a contradiction.

Suppose that the algorithm refuses to end a phase. Without loss of generality, during some phase, the algorithm keeps $p_A \geq \varepsilon$. After the adversary requests B for M steps, the algorithm pays at least $M\varepsilon$, while the adversary pays nothing, again yielding a contradiction if M is large. □