Computer Science 789 Fall 1999

1 Proof Techniques

1.1 Vocabulary

function

predicate

tautology

contradiction

satisfiable

1.2 Boolean as an ordered type

0 false

1 true

$0 < 1$

$\leq$ implies

$\geq$ if

"if $x \geq 2$ then $x^2 \geq 4$" can be rewritten as "$(x \geq 2) \leq \left(x^2 \geq 4\right)$"

1.3 Quantifiers

$\exists$

$\forall$

max

min

max is a generalization of $\exists$, and min is a generalization of $\forall$.

2 Proving statements involving quantifiers

P is a predicate on a set X, and $F : X \to \mathbb{R}$.

To prove $\exists(x \in X)P(x)$, must find some $a \in X$ for which $P(a)$

To prove $\forall(x \in X)P(x)$, must show that $P(a)$ for all $a \in X$.

To prove $\max_{x \in X} F(x) = K$, must show

$$\forall(x \in X)\{F(x) \leq K\}$$
$$\exists(x \in X)\{F(x) \geq K\}$$

To prove $\min_{x \in X} F(x) = K$, must show

To prove $\max_{x \in X} F(x) \leq K$, must show

$$\forall(x \in X)\{F(x) \leq K\}$$

To prove $\max_{x \in X} F(x) \geq K$, must show

$$\exists(x \in X)\{F(x) \geq K\}$$

Example: prove that $\forall(x \geq 0)\{\max\{x^2 - 1, 0\} \geq x - 1\}$

Break into cases.

3 Adversarial Proofs of Statements involving Quantifiers

Suppose $F, G, H : X \to \mathbb{R}$. Prove that

$$\max_{x \in X} \{F(x) - H(x)\} \leq \max_{x \in X} \{F(x) - G(x)\} + \max_{x \in X} \{G(x) - H(x)\}$$

Step 1 of proof: allow expansion of scopes by using unique variable names.

Step 2 of proof: Allow the adversary to make an arbitrary choice of the "adversarial" variables.

Step 3 of proof: We choose "our" variables, using the adversarial variables as needed.

Steps 2 and 3 can be repeated any number of times. (But not in this simple example.)

Step 4. Verify.

4 Nested Quantifiers

"Some people can't understand any statement with two quantifiers in it."

"What?"

"See what I mean?"

Prove that

$$\max_{x \in X} \min_{y \in Y} F(x, y) \leq \min_{y \in Y} \max_{x \in X} F(x, y)$$

5 Dynamic Programming

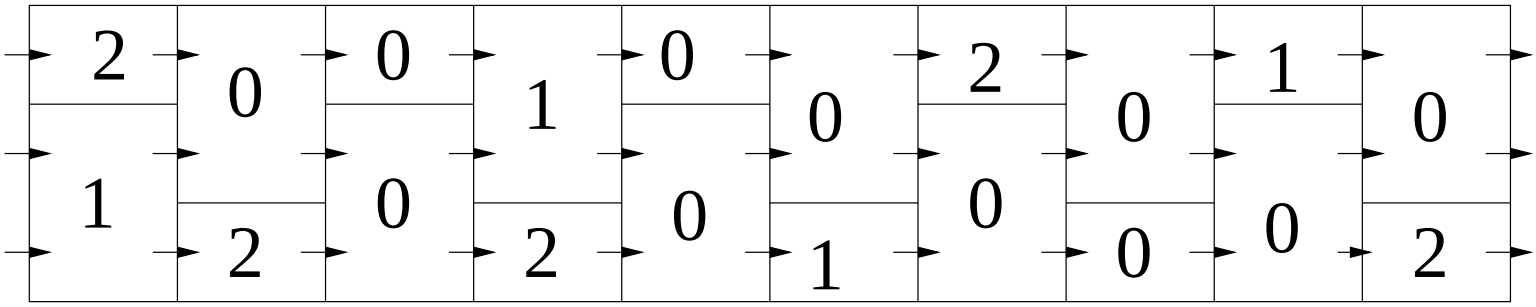


Figure 1: The Layered Dungeon Problem

The greedy strategy yields a treasure of 6, while the optimal path yields a treasure of 9.

6 An Online Problem

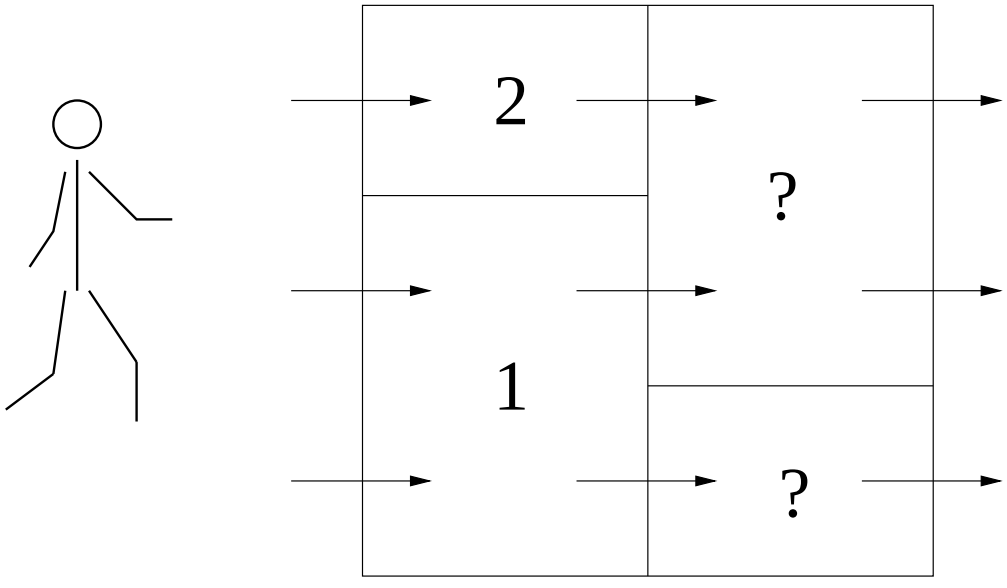


Figure 2: The Online Layered Dungeon Problem