6 Potentials

Suppose a function $F(t)$ is defined for all integers in the range $0 \leq t \leq n$, and we wish to prove

$$F(n) \geq 0$$

One method we might use is *induction*, either the regular or the strong form. To use regular induction, for example, we would need to show

$$\begin{aligned} F(0) &\geq 0 \\ \forall\{t > 0\}\{F(t-1) \geq 0 &\implies F(t) \geq 0\} \end{aligned}$$

To use strong induction, we would need to show

$$\forall\{t \geq 0\}\{\forall\{0 \leq s < t\}\{F(s) \geq 0\} \implies F(t) \geq 0\}$$

In many cases that arise in practice, there is no inductive proof of the statement that $F(t) \geq 0$ using either of the above two formulations. We introduce the *potential* method, which is in fact a sophisticated form of induction.

Lemma 1 *Suppose that there is a function $\Phi(t)$ for all $0 \leq t \leq n$ such that*

$$\begin{aligned} \Phi(n) &\geq 0 \\ F(0) &\geq \Phi(0) \\ \forall(t > 0)\{F(t) - F(t-1) &\geq \Phi(t) - \Phi(t-1)\} \end{aligned}$$

then $F(n) \geq 0$.

Proof: We use collapsing series.

$$\begin{aligned} \sum_{t=1}^{n} \{F(t) - F(t-1)\} &\geq \sum_{t=1}^{n} \{\Phi(t) - \Phi(t-1)\} \\ F(n) - F(0) &\geq \Phi(n) - \Phi(0) \\ F(n) &\geq \Phi(n) \\ &\geq 0 \end{aligned}$$

□

6.1 Graph Potentials

Let $G = (V, E, w)$ be any weighted acyclic directed graph. that is, V is the set of nodes, E is the set of edges, and $w : E \to \mathbf{R}$ is a *weight* function. Recall, from the usual definition of a directed graph, that V is finite.

One classic problem is to compute the minimum or maximum weight path between two nodes in G. Another classic problem is to determine whether G has a negative weight cycle.

We define a *potential* for a weighted directed graph to be real-valued function Ψ on the nodes of the graph, such that, if there is an edge from any node x to a node y, then $\Psi(x) + w(x, y) \geq \Psi(y)$.

Lemma 2 *If x, y are nodes of G, and if G has a potential Ψ, and if α is a path from x to y, then $w(\alpha) \geq \Psi(y) - \Psi(x)$.*

Proof: Let $\alpha = (x_0, x_1, \ldots x_n)$, where $x_0 = x$ and $y = x_n$.

$$\begin{aligned} w(\alpha) &= \sum_{t=1}^{n} w(x_{t-1}, x_t) \\ &\geq \textstyle\sum_{t=1}^{n} \{\Psi(x_t) - \Psi(x_{t-1})\} \\ &= \Psi(y) - \Psi(x) \end{aligned}$$

□

Lemma 3 *G has a negative cycle if and only if G does not have a potential.*

Proof: One direction is an immediate corollary of the previous lemma. Lemma 2. If Ψ is a potential, and α is a cycle, let x be any node in α. We think of α as a path from x to x. By Lemma 2, $w(\alpha) \geq \Psi(x) - \Psi(x) = 0$.

Conversely, suppose that there is no cycle in G of negative weight. For each node x, define $\Psi(x)$ to be the minimum weight of any simple path that ends at x. There are only finitely many such paths, so that minimum exists.

We claim that if α is any path ending at x, then $w(\alpha) \geq \Psi(x)$. The claim is trivial if α is simple. If α is not simple, it must cross itself, at a node, say y. Let β be the "short-circuit" path obtained from α by excising the loop at y. Since the weight of that loop cannot be negative, $w(\beta) \leq w(\alpha)$. Repeat this process until a simple path γ is obtained; γ will also end at x. Thus $w(\alpha) \geq w(\gamma) \geq \Psi(x)$.

We now prove that Ψ is a potential. Let (x, y) be any edge of G. Let α be a path whose weight is $\Psi(x)$. Let β be the path consisting of α followed by the edge (x, y). By the definitions and by the claim

$$\begin{aligned} \Psi(x) + w(x,y) &= w(\alpha) + w(x,y) \\ &= w(\beta) \\ &\geq \Psi(y) \end{aligned}$$

and we are done. □