Distribution of a Sequence

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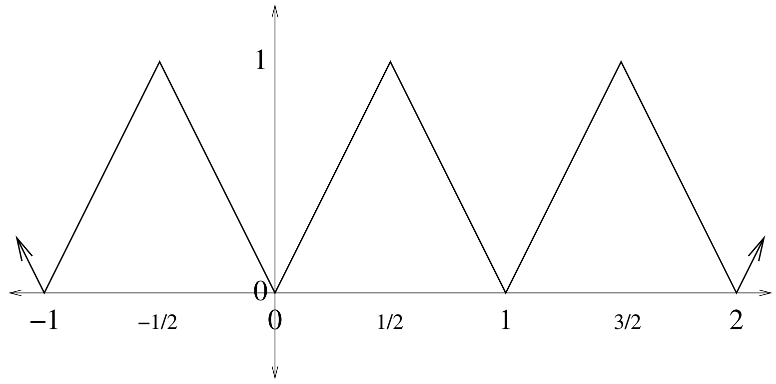
For any real number x, define $round(x)$ to be the integer nearest to x. If x is an odd multiple of $\frac{1}{2}$, we let $round(x) = x - \frac{1}{2}$.

Define $f(x) = 2|round(x) - x|$. We illustrate f in the figure below.



We now use randomization to be a pick a sequence $\mathbf{x} = x_0, x_1, \ldots$ as follows:

1. $0 \leq x_0 < 1$ is chosen uniformly at random.[1]
2. For each $i \geq 1$, $x_i = f(x_{i-1})$.

Problem: what is the long-run behavior of the sequence $\mathbf{x}$? Here are some hand calculations.

- If $x_0 = \frac{1}{4}$, then $\mathbf{x}$ converges to zero: $\mathbf{x} = \frac{1}{4}, \frac{1}{2}, 0, 0, 0, \ldots$
- If $x_0 = \frac{1}{3}$, then $\mathbf{x}$ converges to $\frac{2}{3}$: $\mathbf{x} = \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \ldots$
- If $x_0 = \frac{1}{5}$, then $\mathbf{x}$ does not converge, but rather eventually alternates between two values: $\mathbf{x} = \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \ldots$

However, these choices are obviously very special. What happens in most cases?

[1] This means that, for any $0 \leq a \leq b < 1$, the probability that $a \leq x_0 < b$ is exactly $b - a$.